THETA FUNCTIONS ON VARIETIES WITH EFFECTIVE ANTI-CANONICAL CLASS

MARK GROSS, PAUL HACKING, AND BERND SIEBERT

Abstract. We show that a large class of maximally degenerating families of n-dimensional polarized varieties comes with a canonical basis of sections of powers of the ample line bundle. The families considered are obtained by smoothing a reducible union of toric varieties governed by a wall structure on a real n-(pseudo-)manifold. Wall structures have previously been constructed inductively for cases with locally rigid singularities [GrSi4] and by Gromov-Witten theory for mirrors of log Calabi-Yau surfaces and $K3$ surfaces [GHK1],[GHKS]. For trivial wall structures on the n-torus we retrieve the classical theta functions.

We anticipate that wall structures can be constructed quite generally from maximal degenerations. The construction given here then provides the homogeneous coordinate ring of the mirror degeneration along with a canonical basis. The appearance of a canonical basis of sections for certain degenerations points towards a good compactification of moduli of certain polarized varieties via stable pairs, generalizing the picture for K3 [GHKS]. Another possible application apart from mirror symmetry may be to geometric quantization of varieties with effective anti-canonical class.

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Introduction

It is anticipated that one can construct a mirror to any maximally unipotent degeneration of Calabi-Yau varieties. Precisely, given a Calabi-Yau variety $\mathcal{Y}_\eta$ over $\eta := \operatorname{Spec} \Bbbk((t))$ with maximally unipotent monodromy, one should be able to construct a mirror variety $\mathfrak{X}$ defined over something like the field of fractions of a completion of $\Bbbk[\mathrm{NE}(\mathcal{Y}_\eta)]$, where $\mathrm{NE}(\mathcal{Y}_\eta)$ denotes the monoid of effective curve classes on $\mathcal{Y}_\eta$.

While this general goal has not yet been achieved, various combinations of the authors of this paper have obtained partial results in this direction. For these results a crucial input is a suitably chosen extension $\mathcal{Y} \to \operatorname{Spec} \Bbbk[\![t]\!]$ of $\mathcal{Y}_\eta$.

Starting in [GrSi2] and culminating in [GrSi4], the first and last authors of this paper showed how to construct the mirror if $\mathcal{Y} \to \operatorname{Spec} \Bbbk[\![t]\!]$ was a sufficiently nice polarized *toric degeneration*. This is a degeneration whose central fibre is toric and is described torically near the deepest points of the central fibre. The mirror was then constructed as a toric degeneration $\mathfrak{X} \to \operatorname{Spec} \Bbbk[\![t]\!]$ (and more generally a family of such). The class of toric degenerations is a natural one from the point of view of mirror symmetry, as the mirror of a toric degeneration is a toric degeneration. This point of view incorporates, for example, all Batyrev-Borisov mirrors [Gr1]. However, it is not clear how generally toric degenerations can be constructed given $\mathfrak{Y}_\eta$.

On the other hand, in [GHK1], the first three authors generalized certain aspects of the construction of [GrSi4] to construct the mirror to an arbitrary log Calabi-Yau surface (Y, D) with D an anti-canonical cycle of n rational curves. These mirrors were constructed as smoothings of a union of n copies of $\mathbb{A}^2$, called the n-vertex, and the natural base space for the smoothing is $\operatorname{Spec}(\Bbbk[\![\mathrm{NE}(Y)]\!])$, with $\mathrm{NE}(Y)$ the cone of effective curves in Y. Using similar techniques, [GHKS] will provide mirrors to K3 surfaces $\mathfrak{Y}_\eta$. This construction in particular will provide canonical families over certain toroidal compactifications of F_g, the moduli space of K3 surfaces of genus g. Both these papers used *theta functions*, certain canonically defined functions, as a key part of the construction. In particular, in [GHK1], while it was easy to describe deformations of the n-vertex with origin deleted, theta functions were necessary to provide an extension of such deformations across the origin.

Nevertheless, the key point in common to these constructions is an explicit description of the family $\mathfrak{X}$ whose starting point is combinatorial data recorded in a cell complex of integral convex polyhedra that form a topological manifold B, along with some additional data. The family is then constructed by patching standard toric pieces extracted from the discrete data with corrections carried by a *wall structure*, a collection of real codimension one rational polyhedra along with certain polynomial data. The main difficulty is then determining a suitable wall structure. In the case of [GrSi4], the wall structure was determined by a small amount of additional polynomial starting data, and then an inductive process for the k-th step determined the family $\mathfrak{X} \to \operatorname{Spec} \Bbbk[\![t]\!]$ modulo t^{k+1}. In [GHK1], however, the wall structure was written down all at once in terms of enumerative data on (Y, D). The wall structure in this case records the Gromov-Witten theory of so-called $\mathbb{A}^1$-curves in $Y \setminus D$, rational curves meeting D in exactly one point. In some sense it is a limiting case of [GrSi4] in that the one-parameter families of [GrSi4] arise after certain localizations of the base. In fact, the core argument for why the wall structures provides a well-defined deformation relies on the reduction to this situation via the enumerative interpretation of the inductive process of wall insertion for toric surfaces in [GPS]. A combination of the two methods will be used in [GHKS] in the general K3 case.

The purpose of this paper is two-fold. First, we want to provide a unified framework for both kinds of construction. The focus is not on the specific construction of wall structures that, depending on context, can come from an inductive insertion process as in [GrSi4] or be related to certain enumerative invariants as in [GHK1]. Rather we are striving for maximal generality in the treatment of singularities allowed on B and in the treatment of parameters leading to higher dimensional base spaces of families. The general construction of wall-crossing structures will be done elsewhere using the framework developed in this paper.

Second, and more importantly, we treat in this framework the occurrence of a canonical basis of global functions or, in the projective setting, of sections of powers of an ample line bundle that the construction comes with. Thus we provide here, in the projective setting, the homogeneous coordinate ring of the family via an explicit basis as a module over the base space. In the case of degenerations of abelian varieties the canonical sections agree with classical theta functions. In fact, in Section 6 we show that we obtain all classical theta functions naturally within our framework. We thus also call our canonical sections *theta functions*.

One of the main results of this paper is therefore the existence of theta functions in the canonical degenerations constructed in [GrSi4].

Theorem 0.1. *Let $\pi : \mathfrak{X} \to S$ be one of the canonical degenerations of varieties with effective anticanonical bundle over a complete local ring S constructed in* [GrSi4]. *Assume that there is an ample line bundle L on the central fibre $X_0 \subseteq \mathfrak{X}$ that restricts to the natural ample line bundles on the irreducible components provided by the construction.*

Then there is a distinguished extension of L to an ample line bundle $\mathfrak{L}$ on $\mathfrak{X}$, and $\mathfrak{L}^d$ for $d \geq 1$ has a canonical basis of sections indexed by the $1/d$-integral points of B, the integral affine manifold underlying the construction.

In the appendix we clarify the natural parameter space S for [GrSi4] in the Calabi-Yau situation, under the natural local indecomposability assumption of the discrete data ("simple singularities"). Theorem 0.1 then follows from the principal technical result Theorem 4.12.

In somewhat more detail, we discuss the broad picture presented in this paper. The fundamental combinatorial object of the construction is an integral affine manifold with singularities B with a polyhedral decomposition $\mathscr{P}$. The singular locus is taken to be as large as is possible for our approach: it is (modulo some issues along the boundary of B) the union of codimension two cells of the barycentric subdivision of $\mathscr{P}$ not intersecting the interiors of maximal cells. Thus the singular locus is considerably bigger than is taken in [GrSi4]. Further, unlike the previously cited work, we don't actually insist that B is a manifold: it can fail to be a manifold in codimension ≥ 3. While we do not give the details here, a typical situation in which such a B arises is as the dual intersection complex of a dlt minimal model of a maximally unipotent degeneration of Calabi-Yau varieties. Kollár and Xu in [KoXu], §33 showed that B will indeed be a manifold off of a codimension three subset (see also [NX] for somewhat weaker results).

The parameterizing family S for our construction then arises by choosing a ring A and a toric monoid Q, so that we take $S = \operatorname{Spec} A[Q]/I$ for various choices of ideal I with radical a fixed ideal I_0. In [GrSi4], Q was taken to be $\mathbb{N}$, while in [GHK1], typically Q was closely related to $\mathrm{NE}(Y)$. An additional combinatorial piece of data is a multi-valued piecewise linear function φ defined on $B_0 := B \setminus \Delta$ with values in $Q^{\mathrm{gp}} \otimes_{\mathbb{Z}} \mathbb{R}$. In [GrSi4], this is viewed as specified data, while in [GHK1], this function is canonically given by the mirror construction presented there. This combinatorial data is all described in §1.

The goal then is to specify additional information which determines an appropriate family $\mathfrak{X} \to S$. This family should have the property that $\mathfrak{X} \times_S \operatorname{Spec} A[Q]/I_0$ is a union of polarized toric varieties defined over $\operatorname{Spec} A[Q]/I_0$; these polarized toric varieties are determined by their Newton polyhedra, which run over the maximal cells of $\mathscr{P}$, and are glued together as dictated by the combinatorics of $\mathscr{P}$. The local structure of this family over S in neighbourhoods of codimension one strata should roughly be determined by the function φ.

The necessary additional information is a *wall structure* $\mathscr{S}$, consisting of a collection of walls with attached functions. These walls instruct us how to specify gluings between various standard charts. However, unlike in [GrSi4], we only have models for charts in codimensions 0 and 1, and thus a wall structure is only able to produce a thickening $\mathfrak{X}^\circ \to S$ of $X_0^\circ \to \operatorname{Spec} A[Q]/I_0$, the reduced scheme obtained from X_0 by deleting codimension ≥ 2 strata. This construction is explained in §2.

Roughly, in a mirror symmetry context, a wall structure can be viewed as a way of encoding information about Maslov index zero disks with boundary in the fibre of an SYZ fibration (where B plays the role of the base of the fibration). We expect, based on our experiences in [GHK1] and [GHKS], that it will be possible to define suitable wall structures in great generality using a version of logarithmic Gromov-Witten invariants which shall be presented in forthcoming work of Abramovich, Chen, Gross and Siebert [ACGS].

This leaves the question of (partially) compactifying the family $\mathfrak{X}^\circ \to S$ to $\mathfrak{X} \to S$. This is where we make contact with the innovation of [GHK1], where theta functions were used precisely to achieve this compactification. If X_0 is affine, then a flat infinitesimal deformation will also be affine, and hence we can hope to construct $\mathfrak{X}$ by taking the spectrum of the $A[Q]/I$-algebra $\Gamma(\mathfrak{X}^\circ, \mathcal{O}_{\mathfrak{X}^\circ})$. Thus we need the latter algebra to be sufficiently large. This is achieved via the general construction of theta functions, given in §3, using broken lines. These were introduced in [Gr2] and first used to construct regular functions in the context of [GrSi4] in [CPS]. The definition of broken line depends on the structure $\mathscr{S}$, and we say a structure is *consistent* if suitable counts of broken lines yield regular functions on $\mathfrak{X}^\circ$. In the consistent, affine case, theta functions can then be viewed as canonically given lifts of monomial functions on X_0, and they are labelled by asymptotic directions on B.

So far, this only allows the partial compactification in the affine case. However, in §4, we turn to the general case, most importantly including the projective case. The key point is that in general we can reduce to the affine case as follows. The cone over B itself carries a natural affine structure, and corresponds (after suitably truncating the cone) to the total space of $\mathfrak{L}^{-1}$, where $\mathfrak{L}$ is an ample line bundle on $\mathfrak{X}^\circ$ specified by the data of B. Then regular functions on the total space of $\mathfrak{L}^{-1}$ homogeneous of weight $d \geq 0$ with respect to the fibrewise $\mathbb{G}_m$-action correspond to sections of $\mathfrak{L}^{\otimes d}$. As a result, one is able to construct a homogeneous coordinate ring for $\mathfrak{X}$.

In this projective context, theta functions are then viewed as sections of $\mathfrak{L}^{\otimes d}$ for $d \geq 0$, and if $d > 0$, these functions are parameterized by the set $B(\frac{1}{d}\mathbb{Z})$, the set of points of B with coordinates in $\frac{1}{d}\mathbb{Z}$. Broken lines can then be viewed via projection from the truncated cone over B to B, to obtain objects we call jagged paths, see §4.5. In fact, historically, jagged paths were discovered before broken lines, in discussions between the first and fourth authors of this paper and Mohammed Abouzaid.

The construction is then summarized as follows in the case that B is compact. For a given base ring $R = A[Q]/I$, we define a homogeneous graded R-algebra

$$A := R \oplus \bigoplus_{d>0} \bigoplus_{p \in B(\frac{1}{d}\mathbb{Z})} R\vartheta_p.$$

We give a tropical rule for the multiplication law in terms of counting trees with three leaves where the edges are jagged paths, or the corresponding count in terms of broken lines on the truncated cone over B. Note that here associativity follows from the fact that the functions ϑ_p are actually functions on $\mathfrak{X}^\circ$ constructed by gluing. We then define $\mathfrak{X} = \operatorname{Proj} A$.

Thus we emphasize there are three levels of tropical constructions: the wall structure $\mathscr{S}$ on B which governs the construction can be viewed as a tropicalization of Maslov index zero disks. In the affine case the broken lines which describe theta functions can be viewed as a tropicalization of Maslov index two disks, while the trees which yield the multiplication law can be viewed as a tropicalization of Maslov index four disks. In the projective case, jagged paths contributing to the description of a theta function can be viewed as tropicalizations of holomorphic disks contributing to Floer multiplication for two Lagrangian sections and a fibre of the SYZ fibration, while the trees which yield multiplication can be viewed as a tropicalization of holomorphic disks contributing to Floer multiplication involving three Lagrangian sections. This latter point of view has been explained in [DBr], Chapter 8 and [GrSi6].

Once a suitable theory for counting such disks in an algebro-geometric setting is developed, it should be possible to write down the algebra A directly from enumerative geometry of a log Calabi-Yau variety, generalizing the construction of [GHK1]. This in turn should lead to a general mirror construction for a maximally unipotent family of Calabi-Yau varieties. This chain of ideas will be pursued elsewhere. However, one should view the wall structure $\mathscr{S}$ as giving the richest description of the construction.

The correspondence between points of $B(\frac{1}{d}\mathbb{Z})$ and theta functions is particularly illuminating in the case of abelian varieties. Classically, the existence of a canonical basis of sections of powers of the ample line bundle relies on explicit formulas. In the case of abelian varieties our formal family is the completion of an analytic family $\mathcal{X} \to \tilde{S}$ over an analytic open subset $\tilde{S}$ of an affine toric variety, with $\mathfrak{L}$ the completion of a holomorphic line bundle $\mathcal{L}$. The affine manifold is a real n-torus $B = \mathbb{R}^n/\Gamma$ with $\Gamma \subseteq \mathbb{R}^n$ a lattice of rank n, and $B(\frac{1}{d}\mathbb{Z})$ can be viewed as one-half of the kernel of the polarization induced by $\mathcal{L}^{\otimes d}$. In §6, we then show that our theta functions coincide with classical theta functions. This was the original motivation for using the term "theta function" for our canonical functions.

In the appendix, we make the connection between the general framework we consider here and that of [GrSi4]. We leave this discussion to the appendix as the presentation in the rest of the paper is self-contained, but the appendix relies on greater details from earlier work of the Gross-Siebert program. The discussion of the appendix leads to the proof of Theorem 0.1.

In [GHKK] theta functions are used to construct canonical bases of cluster algebras. A cluster algebra can be understood as the ring of global functions on the interior $U = Y \setminus D$ of a log Calabi–Yau variety (Y, D). The variety U admits a flat degeneration to an algebraic torus which is used to give a perturbative construction of the theta functions. This is a special case of the general construction described in this paper. In the dimension 2 case the theta functions can be described explicitly, see [CZZ]. For cluster varieties describing the open double Bruhat cell in a semi-simple algebraic group it is an open question if our theta functions coincide with Lusztig's canonical basis, see [GHKK], Corollary 0.20 and the discussion following it.

Other cases with an alternative characterization of theta functions include mirrors to certain log Calabi-Yau surfaces [GHK2] and possibly also higher-dimensional Fano varieties with anticanonical polarization.

While in general we do not currently have a characterization of theta functions other than via our construction, the case of abelian varieties does lead to some speculation. Indeed, there is an interpretation of classical theta functions in terms of geometric quantization that also generalizes to moduli spaces of flat bundles over a Riemann surface [APW], [Ht], [Ty], [BMN]. From this point of view, the degeneration of abelian varieties is viewed as a degenerating family of complex structures on a fixed Lagrangian fibration $A \to B$. Similarly, $\mathcal{L}$ can be viewed as a degenerating family of compatible complex structures on a complex line bundle L over A. Viewing the complex structure as a distribution on the tangent spaces, the limit $s \to 0$ is given by the tangent spaces to fibres of the Lagrangian fibration. In this picture, a $1/d$-integral point $x \in B$ labels a distributional section of L with support the fibre of the Lagrangian fibration over x. These distributional sections provide the initial data for the heat equation fulfilled by classical theta functions due to the functional equation.

A similar picture is expected to hold in much greater generality [An]. In the context of geometric quantization of Calabi-Yau varieties with a Lagrangian fibration provided by the SYZ conjecture, the existence of generalized theta functions was indeed conjectured by the late Andrei Tyurin [Ty]. We believe that our theta functions should also fulfill some heat equation with distributional limit over the limiting Lagrangian fibration, but the nature of this equation is unknown to date.

The present theta functions were conjectured to exist by the first and fourth authors of this article in the context of homological mirror symmetry applied to the degenerations of [GrSi4]. In the affine case the first proof of existence in dimension two has appeared in [GHK1], while [CPS] established the existence of canonical functions in any dimension in the framework of [GrSi4]. See [GrSi6] for more details on the history.

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1. The affine geometry of the construction

1.1. **Polyhedral affine pseudomanifolds.** We give a common setup for [GrSi2], [GrSi4] and [GHK1]. An *affine manifold* B_0 is a differentiable manifold with an equivalence class of charts with transition functions in $\mathrm{Aff}(\mathbb{R}^n) = \mathbb{R}^n \rtimes \mathrm{GL}(\mathbb{R}^n)$. It is *integral* if the transition functions lie in $\mathrm{Aff}(\mathbb{Z}^n) = \mathbb{Z}^n \rtimes \mathrm{GL}(\mathbb{Z}^n)$. A map between (integral) affine manifolds preserving this structure is called an *(integral)* *affine map*. In the integral case it makes sense to talk about *$1/d$-integral points* $B_0(\frac{1}{d}\mathbb{Z}) \subseteq B_0$, locally defined as the preimage of $\frac{1}{d}\mathbb{Z}^n \subseteq \mathbb{R}^n$ in a chart. An integral affine manifold B_0 comes with a sheaf of integral (co-) tangent vectors $\Lambda = \Lambda_{B_0}$ (dually $\check{\Lambda} = \check{\Lambda}_{B_0}$) and of integral affine functions $\mathcal{A}\!f\!f(B_0, \mathbb{Z})$. These sheaves are locally constant with stalks isomorphic to $\mathbb{Z}^n$ and to $\mathrm{Aff}(\mathbb{Z}^n, \mathbb{Z}) \simeq \mathbb{Z}^n \oplus \mathbb{Z}$, respectively. The corresponding real versions are denoted $\Lambda_\mathbb{R}$, $\check{\Lambda}_\mathbb{R}$ and $\mathcal{A}\!f\!f(B_0, \mathbb{R})$. Generally, if A is an abelian group then $A_\mathbb{R} := A \otimes_\mathbb{Z} \mathbb{R}$. We have an exact sequence

$$0 \longrightarrow \underline{\mathbb{Z}} \longrightarrow \mathcal{A}\!f\!f(B_0, \mathbb{Z}) \longrightarrow \check{\Lambda} \longrightarrow 0, \tag{1.1}$$

dividing out the constant functions. Taking $\mathrm{Hom}_{B_0}(\check{\Lambda}, \,.\,)$ provides a connecting homomorphism

$$\mathrm{Hom}_{B_0}(\check{\Lambda}, \check{\Lambda}) \longrightarrow \mathrm{Ext}^1_{B_0}(\check{\Lambda}, \mathbb{Z}) = H^1(B_0, \Lambda).$$

The image of the identity defines the *radiance obstruction* of B_0, which is an obstruction class to the existence of a set of charts with linear rather than affine transition functions (see [GH] or [GrSi2], pp.179ff).

A (convex) *polyhedron* in $\mathbb{R}^n$ is the solution set of finitely many affine inequalities. A polyhedron is *integral* if each face contains an integral point and the affine inequalities can be taken with rational coefficients. In particular, any vertex of an integral polyhedron is integral. We use lower case Greek letters for integral polyhedra, where we reserve $\sigma, \sigma', \ldots$ for maximal cells and ρ for codimension-one cells. For a polyhedron τ we write $\partial\tau$ for the union of proper faces of τ and $\mathrm{Int}\,\tau := \tau \setminus \partial\tau$ for the complement. Note that for $\tau \subseteq \mathbb{R}^n$ and $\dim\tau < n$ this does not agree with the topological boundary. Another notation is Λ_τ for the sheaf of integral tangent vectors on τ, viewed as an integral affine manifold with boundary. We will not be too picky and sometimes also use the notation Λ_τ for the stalk of Λ_τ at any $y \in \mathrm{Int}\,\tau$ or the abelian group of global sections $\Gamma(\tau, \Lambda_\tau)$. The precise meaning should always be obvious from the context. Also, if $\tau \subseteq \tau'$ we consider Λ_τ naturally as a subgroup of $\Lambda_{\tau'}$.

The arena for all that follows is a topological space B of dimension n, possibly with boundary, with an integral affine structure on $B_0 := B \setminus \Delta$ with $\Delta \subseteq B$ of codimension two, and a compatible decomposition $\mathscr{P}$ into integral polyhedra. Unlike in much previous work, we will not assume that B is a manifold, but rather will have some weaker properties. The details are contained in the following construction.

Construction 1.1. *(Polyhedral affine manifolds.)* Let $\mathscr{P}$ be a set of integral polyhedra along with a set of integral affine maps $\omega \to \tau$ identifying ω with a face of τ, making $\mathscr{P}$ into a category. We require that any proper face of any $\tau \in \mathscr{P}$ occurs as the domain of an element of $\mathrm{hom}(\mathscr{P})$ with target τ. We assume that the direct limit in the category of topological spaces

$$B := \varinjlim_{\tau \in \mathscr{P}} \tau$$

satisfies the following conditions:

(1) For each $\tau \in \mathscr{P}$ the map $\tau \to B$ is injective, that is, no cells self-intersect (unlike in [GrSi2]).
(2) By abuse of notation we view the elements of $\mathscr{P}$ as subsets of B, also referred to as *cells* of $\mathscr{P}$. We assume that the intersection of any two cells of $\mathscr{P}$ is a cell of $\mathscr{P}$.
(3) B is pure dimension n, in the sense that every cell of $\mathscr{P}$ is contained in at least one n-dimensional cell.
(4) Every $(n-1)$-dimensional cell of $\mathscr{P}$ is contained in one or two n-dimensional cells, so that B is a manifold with boundary away from codimension ≥ 2 cells.
(5) *The S_2 condition.* If $\tau \in \mathscr{P}$ satisfies $\dim \tau \leq n-2$, then for any $x \in \operatorname{Int} \tau$ and any open neighbourhood $U \subseteq B$ of x, there is a connected open neighbourhood $V \subseteq U$ of x such that $V \setminus \tau$ is also connected.

If $\mathscr{P}$ consisted only of simplices, then the above conditions are somewhat stronger than the usual notion of pseudomanifold (with boundary). For lack of better terminology, and to remind the reader that B need not be a manifold, we call B a pseudomanifold, but the reader should also remember the precise conditions stated above.

Cells of dimensions 0, 1 and n are also called *vertices*, *edges* and *maximal cells.* The notation for the set of k-cells is $\mathscr{P}^{[k]}$ and we often write $\mathscr{P}_{\max} := \mathscr{P}^{[n]}$ for the set of maximal cells. A cell $\rho \in \mathscr{P}^{[n-1]}$ only contained in one maximal cell is said to lie on the *boundary* of B, and we let ∂B be the union of all $(n-1)$-cells lying on the boundary of B. Any cell of $\mathscr{P}$ contained in ∂B is called a *boundary cell.* Cells not contained in ∂B are called *interior*, defining $\mathscr{P}_{\mathrm{int}} \subseteq \mathscr{P}$. Thus $\mathscr{P}_\partial := \mathscr{P} \setminus \mathscr{P}_{\mathrm{int}}$ is the induced polyhedral decomposition of ∂B.

Next we want to endow B with an affine structure outside a subset $\Delta \subseteq B$ of codimension two, sometimes referred to as the *discriminant locus.* For Δ we take the union of the $(n-2)$-cells of a barycentric subdivision $\tilde{\mathscr{P}}$ of $\mathscr{P}$ that neither intersect the interiors of maximal cells nor the interiors of maximal cells of the boundary ∂B. Two remarks are in order here. First, while the barycenter of a bounded polyhedron can be defined invariantly in affine geometry, the precise location of Δ is not important as long as it respects the cell structure. So the construction of Δ is purely topological. Second, for an unbounded cell τ we take the barycenter at infinity, that is, replace the barycenter by an unbounded direction $u_\tau \in (\Lambda_\tau)_\mathbb{R}$. A piecewise linear choice of Δ is explained in [GrSi4], p.1310 and runs as follows. For each bounded cell choose a point $a_\tau \in \operatorname{Int}\tau$, which is to become its barycenter. For unbounded cells the direction vectors u_τ need to be parallel for faces with the same asymptotic cone.[1] Then a k-cell of $\tilde{\mathscr{P}}$ labelled by a sequence $\tau_0 \subsetneq \tau_1 \subsetneq \ldots \subsetneq \tau_k$ in $\mathscr{P}$ with $\tau_0, \ldots, \tau_l$, $l \ge 0$, bounded and $\tau_{l+1}, \ldots, \tau_k$ unbounded is taken as $\operatorname{conv}\{a_{\tau_0}, \ldots, a_{\tau_l}\} + \sum_{i=l+1}^k \mathbb{R}_{\ge 0} u_{\tau_i}$.[2] For any unbounded $\tau \in \mathscr{P}$ there is then a deformation retraction of $\tau \setminus \Delta$ to the union of bounded faces of $\tau \setminus \Delta$. Note that if $\sigma, \sigma' \in \mathscr{P}_{\max}$ intersect in $\rho \in \mathscr{P}^{[n-1]}$ then $\rho \not\subseteq \partial B$ and $\rho \setminus \Delta$ has a number of connected components, one for each $(n-1)$-cell of the barycentric subdivision of ρ. Thus each connected component of $\rho \setminus \Delta$ is labelled uniquely by a sequence $\tau_0 \subseteq \tau_1 \subseteq \ldots \subseteq \tau_{n-1} = \rho$ with $\tau_k \in \mathscr{P}^{[k]}_{\mathrm{int}}$. We denote such an $(n-1)$-cell of the barycentric subdivision of ρ by $\underline{\rho} \in \tilde{\mathscr{P}}^{[n-1]}_{\mathrm{int}}$. With this notation it is understood that $\underline{\rho}$ is contained in $\rho \in \mathscr{P}^{[n-1]}_{\mathrm{int}}$. In particular, we take only those $(n-1)$-cells of $\tilde{\mathscr{P}}$ that do not intersect the interiors of maximal cells of $\mathscr{P}$.

To define an affine structure on $B_0 := B \setminus \Delta$ compatible with the given affine structure on the cells it suffices to provide, for each $\underline{\rho} \in \tilde{\mathscr{P}}^{[n-1]}_{\mathrm{int}}$, an identification of tangent spaces of the adjacent maximal cells σ, σ' inducing the identity on Λ_ρ. Equivalently, if $\xi \in \Lambda_\sigma$ is such that $\Lambda_\rho + \mathbb{Z}\xi = \Lambda_\sigma$ then for each $\underline{\rho} \in \tilde{\mathscr{P}}^{[n-1]}_{\mathrm{int}}$ with $\underline{\rho} \subseteq \rho$ we have to provide $\xi' \in \Lambda_{\sigma'}$ with $\Lambda_\rho + \mathbb{Z}\xi' = \Lambda_{\sigma'}$. Each such data defines an integral affine structure on B_0.

This ends the construction of the pseudomanifold B, a codimension two subset Δ, a decomposition $\mathscr{P}$ of B into integral affine polyhedra and a compatible integral affine structure on B_0. For brevity we refer to all these data as a *polyhedral affine pseudomanifold* or just *polyhedral pseudomanifold*, denoted $(B, \mathscr{P})$.

Remark 1.2. The complement B_0 of Δ retracts onto a simplicial complex of dimension one. In fact, by the very definition of Δ, B_0 is covered by the interiors of the maximal cells $\sigma \in \mathscr{P}$ and by $\underline{\rho} \setminus \Delta$, $\underline{\rho} \in \tilde{\mathscr{P}}^{[n-1]}_{\mathrm{int}}$. By assumption on B, each interior $(n-1)$-cell is contained in precisely two maximal cells. Thus B_0 deformation retracts to a one-dimensional simplicial subspace having one vertex $a_\sigma \in \operatorname{Int}\sigma$ for each $\sigma \in \mathscr{P}_{\max}$ and an edge connecting a_σ, $a_{\sigma'}$ for each $\underline{\rho} \in \tilde{\mathscr{P}}^{[n-1]}_{\mathrm{int}}$ with $\underline{\rho} \subseteq \sigma \cap \sigma'$.

There are two major series of examples.

Examples 1.3. 1) In [GrSi2], [GrSi4] the affine structure extends over a neighbourhood of the vertices. In fact, in this case we can replace Δ by the union

[1] The notion of asymptotic cone is discussed at the beginning of §2.

[2] Note that in the unbounded case the τ_i need to have strictly ascending asymptotic cones for the dimension of this cell of $\tilde{\mathscr{P}}$ to be k.

$\breve{\Delta}$ of $(n-2)$-cells of Δ not containing any vertex. This example also requires a compatibility condition between the charts ([GrSi4], Definition 1.2), which only arises if B_0 intersects cells of codimension at least two. Additional aspects of the main body of this paper particular to this case are discussed in the appendix. Note in this case B is actually a topological manifold.

2) In [GHK1], [GHKS], the polyhedral affine pseudomanifolds used (while still actually manifolds) are quite different from those of (1). These two papers give two-dimensional examples where all singularities occur at vertices of a polyhedral decomposition. In the case of [GHK1], one starts with a so-called Looijenga pair (Y, D), that is, Y is a rational surface and $D \in |-K_Y|$ is a cycle of rational curves. Write $D = D_1 + \cdots + D_n$ in cyclic order. One associates to the pair its dual intersection complex (B, Σ). Topologically $B = \mathbb{R}^2$ and Σ, the polyhedral decomposition, is a complete fan with a two-dimensional cone $\sigma_{i,i+1}$ associated to each double point $D_i \cap D_{i+1}$ and ray $\rho_i = \sigma_{i-1,i} \cap \sigma_{i,i+1}$ associated to each irreducible component D_i. Abstractly, $\sigma_{i,i+1}$ is integral affine isomorphic to the first quadrant of $\mathbb{R}^2$. The discriminant locus Δ coincides with the zero-dimensional cell in Σ, which we denote by 0. The affine structure on B_0 is given by charts

$$\psi_i : U_i = \mathrm{Int}(\sigma_{i-1,i} \cup \sigma_{i,i+1}) \to \mathbb{R}^2$$

where ψ_i is defined on the closure of U_i by

$$\psi_i(v_{i-1}) = (1, 0), \quad \psi_i(v_i) = (0, 1), \quad \psi_i(v_{i+1}) = (-1, -D_i^2)$$

with v_i denoting a primitive generator of ρ_i and ψ_i is defined linearly on the two two-dimensional cones.

If one wishes a compact example with boundary, one can choose a compact two-dimensional subset $\bar{B} \subseteq B$ with polyhedral boundary and $0 \in \mathrm{Int}\,\bar{B}$. In certain cases one may find such a $\bar{B}$ with locally convex boundary. Indeed, in [GHK2], it is shown that such a $\bar{B}$ exists if and only if D supports a nef and big divisor.

In [GHKS], we will need a version of this applied to degenerations of K3 surfaces. Let $\mathcal{Y} \to T$ be a one-parameter degeneration of K3 surfaces which is simple normal crossings, relatively minimal, and maximally unipotent. Let $(B, \mathscr{P})$ be the dual intersection complex of the degenerate fibre: $\mathscr{P}$ has a vertex v for every irreducible component Y_v of the central fibre $\mathcal{Y}_0$, and $\mathscr{P}$ contains a simplex with vertices $v_0, \ldots, v_n$ if $Y_{v_0} \cap \cdots \cap Y_{v_n} \neq \emptyset$. We take Δ to be the set of vertices. The affine structure is defined as follows. Each two-dimensional simplex of $\mathscr{P}$ carries the affine structure of the standard simplex. Given simplices of $\mathscr{P}$ with a common edge,

$$\sigma_1 = \langle v_0, v_1, v_2 \rangle, \quad \sigma_2 = \langle v_0, v_1, v_3 \rangle,$$

we define a chart $\psi : \mathrm{Int}(\sigma_1 \cup \sigma_2) \to \mathbb{R}^2$ via

(1.2)
$$\psi(v_0) = (0,0), \quad \psi(v_1) = (0,1), \quad \psi(v_2) = (1,0), \quad \psi(v_3) = (-1, -(Y_{v_0} \cap Y_{v_1})^2),$$

where the latter self-intersection is computed in Y_{v_0}. Again, ψ is affine linear on each two-cell.

These constructions generalize to higher dimensions, producing many examples of polyhedral affine pseudomanifolds with singular locus the union of codimension two cells. This can be applied, for example, to log Calabi-Yau manifolds with suitably well-behaved compactifications, or to log smooth relatively minimal maximally unipotent degenerations of Calabi-Yau manifolds. More generally, one can consider relatively minimal dlt models of such degenerations. The general construction will be taken up elsewhere.

Continuing with the general case, an important piece of data that comes with a polyhedral pseudomanifold are certain tangent vectors along any codimension one cell ρ that encode the monodromy of the affine structure in a neighbourhood of ρ. Let $\underline{\rho}, \underline{\rho}' \subseteq \rho$ be two $(n-1)$-cells of the barycentric subdivision, and let $\sigma, \sigma' \in \mathscr{P}^{[n]}$ be the maximal cells adjacent to ρ. Consider the affine parallel transport T along a path starting from $x \in \mathrm{Int}\,\underline{\rho}$ via $\mathrm{Int}\,\sigma$ to $\mathrm{Int}\,\underline{\rho}'$ and back to x through $\mathrm{Int}\,\sigma'$. By the definition of the affine structure on B_0 this transformation leaves $\Lambda_\rho \subseteq \Lambda_x$ invariant. Thus T takes the form

$$T(m) = m + \check{d}_\rho(m) \cdot m_{\underline{\rho}\underline{\rho}'}, \quad m \in \Lambda_x, \tag{1.3}$$

where $\check{d}_\rho \in \check{\Lambda}_x$ is a generator of $\Lambda_\rho^\perp \subseteq \check{\Lambda}_x$ and $m_{\underline{\rho}\underline{\rho}'} \in \Lambda_\rho$. To fix signs we require $\check{d}_\rho$ to take non-negative values on σ. Since changing the roles of σ and σ' reverses both the sign of $\check{d}_\rho$ and the orientation of the path, the *monodromy vector* $m_{\underline{\rho}\underline{\rho}'}$ is well-defined. Note also that $m_{\underline{\rho}'\underline{\rho}} = -m_{\underline{\rho}\underline{\rho}'}$.

In the first series of examples (Example 1.3,1) the connected components of $\rho\backslash\Delta$ are in bijection with vertices $v \in \rho$, and the notation was $m_\rho^{vv'}$. In the second series of examples (Example 1.3,2) the affine structure extends to a neighbourhood of $\mathrm{Int}\,\rho$ and hence $m_{\underline{\rho}\underline{\rho}'} = 0$.

The last topic in this subsection concerns the case $\partial B \neq \emptyset$. First note that the boundary ∂B of B does not generally carry a natural structure of connected polyhedral pseudomanifold. In fact, $\partial B \setminus \Delta$ is merely the disjoint union of the interiors of the cells $\rho \in \mathscr{P}^{[n-1]} \backslash \mathscr{P}^{[n-1]}_{\mathrm{int}}$. An exception is if for any pair of adjacent $(n-1)$-cells $\rho, \rho' \subseteq \partial B$ the tangent spaces Λ_ρ, $\Lambda_{\rho'}$ are parallel, measured in a chart at some point close to $\rho \cap \rho' \in \mathscr{P}^{[n-2]}$. Then ∂B with the induced polyhedral decomposition is naturally a polyhedral sub-pseudomanifold of $(B, \mathscr{P})$. While this case has some special importance (see e.g. [CPS]), it is irrelevant in this paper. We therefore always assume Δ contains the $(n-2)$-skeleton of ∂B.

Unlike in [GrSi4] we also make no assumption on local convexity of B along its boundary.

1.2. **Convex, piecewise affine functions.** The next ingredient is a multi-valued convex PL-function on B_0. Here "PL" stands for "piecewise linear". Let Q be a toric monoid and $Q_{\mathbb{R}} \subseteq Q^{\mathrm{gp}}_{\mathbb{R}}$ the corresponding cone, that is, $Q = Q^{\mathrm{gp}} \cap Q_{\mathbb{R}}$. Recall that a monoid Q is called *toric* if it is finitely generated, integral, saturated and if in addition Q^{gp} is torsion-free. Thus toric monoids are precisely the monoids that are isomorphic to a finitely generated submonoid of a free abelian group.[3]

Definition 1.4. A Q^{gp}*-valued piecewise affine (PA-) function* on an open set $U \subseteq B_0 = B \setminus \Delta$ is a continuous map

$$U \longrightarrow Q^{\mathrm{gp}}_{\mathbb{R}}$$

which restricts to a $Q^{\mathrm{gp}}_{\mathbb{R}}$-valued integral affine function on each maximal cell of $\mathscr{P}$. The sheaf of Q^{gp}-valued integral piecewise affine functions on B_0 is denoted $\mathcal{PA}(B, Q^{\mathrm{gp}})$. The sheaf of Q^{gp}*-valued piecewise linear (PL-) functions* is the quotient $\mathcal{PL}(B, Q^{\mathrm{gp}}) := \mathcal{PA}(B, Q^{\mathrm{gp}})/\underline{Q}^{\mathrm{gp}}$ by the locally constant functions. The respective spaces of global sections are denoted $\mathrm{PA}(B, Q^{\mathrm{gp}})$ and $\mathrm{PL}(B, Q^{\mathrm{gp}})$.

Remark 1.5. This definition is less restrictive than the one given in [GrSi2], Definition 1.43. In particular, we do not require that locally around the interior of $\rho \in \mathscr{P}^{[n-1]}$ a PA-function φ is the sum of an affine function and a PA-function on the quotient fan along ρ. If $\rho \not\subseteq \partial B$ then this quotient fan is just the fan of $\mathbb{P}^1$ in $\mathbb{R}$. The condition says that the change of slope (cf. Definition 1.6 below) of φ along a connected component of $\rho \setminus \Delta$ is independent of the choice of connected component. See Example 1.7 for an illustration.

The change of a PA- (or PL-) function φ along $\underline{\rho} \in \tilde{\mathscr{P}}^{[n-1]}_{\mathrm{int}}$ is given by an element $\kappa \in Q^{\mathrm{gp}}$ as follows. Let σ, σ' be the two maximal cells containing $\underline{\rho}$. Then $V := \operatorname{Int}\sigma \cup \operatorname{Int}\sigma' \cup \operatorname{Int}\underline{\rho}$ is a contractible open neighbourhood of $\operatorname{Int}\underline{\rho}$ in $B_0 = B \setminus \Delta$. An affine chart at $x \in \operatorname{Int}\underline{\rho}$ thus provides an identification $\Lambda_\sigma = \Lambda_x = \Lambda_{\sigma'}$. Let $\delta : \Lambda_x \to \mathbb{Z}$ be the quotient by $\Lambda_\rho \subseteq \Lambda_x$. Fix signs by requiring that δ is non-negative on tangent vectors pointing from ρ into σ'. Let

[3] We do not require toric monoids to be sharp, that is, Q may have non-trivial invertible elements.

$n, n' \in \check{\Lambda}_x$ be the slopes of $\varphi|_\sigma$, $\varphi|_{\sigma'}$, respectively. Then $(n'-n)(\Lambda_\rho) = 0$ and hence there exists $\kappa \in Q^{\mathrm{gp}}$ with

$$n' - n = \delta \cdot \kappa. \tag{1.4}$$

Definition 1.6. The element $\kappa_{\underline{\rho}}(\varphi) := \kappa$ defined in (1.4) is called the *kink* of the Q^{gp}-valued PA-function φ along $\underline{\rho} \in \tilde{\mathscr{P}}^{[n-1]}_{\mathrm{int}}$.

Clearly, a PA-function is integral affine on an open set $U \subseteq B_0$ if and only if $\kappa_{\underline{\rho}}(\varphi) = 0$ whenever $U \cap \underline{\rho} \neq \emptyset$. Moreover, if U is connected, then a PA-function φ on U is determined uniquely by the restriction to $\operatorname{Int}\sigma$ for one $\sigma \in \mathscr{P}_{\max}$ intersecting U and the kinks $\kappa_{\underline{\rho}}(\varphi)$. Conversely, if $U \subseteq B_0$ is simply-connected then there exists a PA-function φ with any prescribed set of kinks $\kappa_{\underline{\rho}}(\varphi) \in Q^{\mathrm{gp}}$.

Example 1.7. To illustrate how the kink can depend on the choice of $\underline{\rho} \subseteq \rho$ let us look at the simplest example of an affine manifold with singularities, see Example 1.16 in [GrSi2], or §3.2 in [GrSi5]. There are only two maximal cells, the 2-simplices $\sigma_1 := \operatorname{conv}\{(-1,0),(0,0),(0,1)\}$, $\sigma_2 := \operatorname{conv}\{(0,0),(1,0),(0,1)\}$. Take $B = \sigma_1 \cup \sigma_2 \subseteq \mathbb{R}^2$ as a topological manifold and $\Delta = \{(0,1/2)\}$ the midpoint of the interior edge ρ. Then $\rho = \underline{\rho}_1 \cup \underline{\rho}_2$ with $\underline{\rho}_\mu \in \tilde{\mathscr{P}}^{[n-1]}_{\mathrm{int}}$ two intervals of integral affine length $1/2$, say $\underline{\rho}_1$ the lower one containing $(0,0)$ and $\underline{\rho}_2$ the upper one containing $(0,1)$. The given embedding into $\mathbb{R}^2$ defines the affine chart on $B \setminus \underline{\rho}_2$ (Chart I). The other chart (Chart II) is given by applying $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ to σ_2. Thus the image of this chart is $\operatorname{conv}\{(-1,0),(0,0),(1,1),(0,1)\}$ minus the image of $\underline{\rho}_1$.

Writing x, y for the standard coordinates on $\mathbb{R}^2$, consider the function φ that in Chart I is given by y. In this chart it is an affine function and hence has kink $\kappa = 0$. However, in Chart II the restriction of φ to the image of σ_1 equals y and the restriction to the image of σ_2 equals $y - x$. Thus $\kappa_{\underline{\rho}_1}(\varphi) = 0$ while $\kappa_{\underline{\rho}_2}(\varphi) = -1$. In particular, φ is not a piecewise affine function in the sense of [GrSi2], Definition 1.43, but it is in the sense of this paper.

Note also that the described phenomenon can only occur under the presence of non-trivial monodromy $m_{\underline{\rho}\underline{\rho}'} \neq 0$ along ρ (1.3).

Definition 1.8. The sheaf of Q^{gp}*-valued multivalued piecewise affine (MPA-) functions* on $B_0 = B \setminus \Delta$ is

$$\mathcal{MPA}(B, Q^{\mathrm{gp}}) := \mathcal{PA}(B, Q^{\mathrm{gp}})/\mathcal{A}\!\mathit{ff}(B, Q^{\mathrm{gp}}).$$

A section of $\mathcal{MPA}(B, Q^{\mathrm{gp}})$ over an open set $U \subseteq B_0$ is called a *(Q^{gp}-valued) MPA-function*, and we write $\mathrm{MPA}(B, Q^{\mathrm{gp}}) := \Gamma(B_0, \mathcal{MPA}(B, Q^{\mathrm{gp}}))$.

Note that dividing out locally constant functions gives the alternative definition

$$\mathcal{MPA}(B, Q^{\mathrm{gp}}) = \mathcal{PL}(B, Q^{\mathrm{gp}})/\mathcal{H}om(\Lambda, \underline{Q}^{\mathrm{gp}}).$$

Since $\mathcal{H}om(\Lambda, \underline{Q}^{\mathrm{gp}}) = \check{\Lambda} \otimes \underline{Q}^{\mathrm{gp}}$ there is an exact sequence of abelian sheaves on $B \setminus \Delta$,

$$0 \longrightarrow \check{\Lambda} \otimes \underline{Q}^{\mathrm{gp}} \longrightarrow \mathcal{PL}(B, Q^{\mathrm{gp}}) \longrightarrow \mathcal{MPA}(B, Q^{\mathrm{gp}}) \longrightarrow 0. \tag{1.5}$$

The connecting homomorphism of the restriction to $U \subseteq B \setminus \Delta$,

$$c_1 : \mathrm{MPA}(U, Q^{\mathrm{gp}}) \longrightarrow H^1(U, \check{\Lambda} \otimes Q^{\mathrm{gp}}), \tag{1.6}$$

measures the obstruction to lifting an MPA-function φ on U to a PL-function. The notation $c_1(\varphi)$ comes from the interpretation on the Legendre-dual side as being the affine representative of the first Chern class of a line bundle defined by φ, see [GrSi2], [GrSi3]. We are working in what is called the *cone picture* here ([GrSi2], §2.1), while the bulk of the discussion in [GrSi2], [GrSi3] takes place in the *fan picture* ([GrSi2],§2.2). The two pictures are related by a *discrete Legendre transform* ([GrSi2], §1.4), which swaps the roles of $c_1(\varphi)$ and the radiance obstruction c_B of B ([GrSi2], Proposition 1.50,3). Thus in the current paper, $c_1(\varphi)$ takes the role of the radiance obstruction in [GrSi3], which represents the residue of the Gauss-Manin connection ([GrSi3], Theorem 5.1,(4)). Hence in the present setup $c_1(\varphi)$ is related to the complex structure moduli. Indeed, the MPA-function φ has a prominent role in the construction of our deformation, see §2.4.

An MPA-function is uniquely determined by its kinks:

Proposition 1.9. *There is a canonical decomposition*

$$\mathcal{MPA}(B, Q^{\mathrm{gp}}) = \bigoplus_{\underline{\rho} \in \tilde{\mathscr{P}}_{\mathrm{int}}^{[n-1]}} Q^{\mathrm{gp}}_{\mathrm{Int}\,\underline{\rho}},$$

where $Q^{\mathrm{gp}}_{\mathrm{Int}\,\underline{\rho}}$ is the push-forward to B_0 of the locally constant sheaf on $\mathrm{Int}\,\underline{\rho}$ with stalks Q^{gp}. The induced canonical isomorphism

$$\mathrm{MPA}(B, Q^{\mathrm{gp}}) = \Gamma\Big(B_0, \textstyle\bigoplus_{\underline{\rho} \in \tilde{\mathscr{P}}_{\mathrm{int}}^{[n-1]}} Q^{\mathrm{gp}}_{\mathrm{Int}\,\underline{\rho}}\Big) = \mathrm{Map}(\tilde{\mathscr{P}}_{\mathrm{int}}^{[n-1]}, Q^{\mathrm{gp}})$$

identifies $\varphi \in \mathrm{MPA}(B, Q^{\mathrm{gp}})$ with the map associating to $\underline{\rho} \in \tilde{\mathscr{P}}_{\mathrm{int}}^{[n-1]}$ the kink $\kappa_{\underline{\rho}}(\varphi) \in Q^{\mathrm{gp}}$ along $\underline{\rho}$ of a local PA-representative of φ.

Proof. This is immediate from the local description (1.4) of piecewise affine functions. □

To obtain local toric models for the deformation construction our MPA-function needs to be convex in the following sense.

Definition 1.10. A *convex (Q-valued) MPA-function* on B is a Q^{gp}-valued MPA-function φ with $\kappa_{\underline{\rho}}(\varphi) \in Q$ for all $\underline{\rho} \in \tilde{\mathscr{P}}_{\mathrm{int}}^{[n-1]}$. The monoid of convex Q-valued MPA-functions on B is denoted $\mathrm{MPA}(B, Q)$.

Example 1.11. 1) In [GrSi2], [GrSi4] we took $Q = \mathbb{N}$ and considered only those functions fulfilling certain additional linear conditions. This defines a subspace of our $\mathrm{MPA}(B, Q^{\mathrm{gp}})$ that can be characterized as follows. The first requirement is $\kappa_{\underline{\rho}}(\varphi) = \kappa_{\underline{\rho}'}(\varphi)$ for any $\underline{\rho}, \underline{\rho}' \in \tilde{\mathscr{P}}^{[n-1]}_{\mathrm{int}}$ contained in the same $(n-1)$-cell ρ of $\mathscr{P}$, see Remark 1.5. We can then write $\kappa_\rho(\varphi)$. The second requirement comes from the behaviour in codimension two. Let $\tau \in \mathscr{P}_{\mathrm{int}}$ be a cell of codimension two and $\rho_1, \ldots, \rho_k$ be the adjacent cells of codimension one. Working in a chart at a vertex $v \in \tau$ let $n_1, \ldots, n_k \in \check{\Lambda}_v$ be the primitive normal vectors to Λ_{ρ_i}, with signs chosen following a simple loop about the origin in $(\Lambda_v)_{\mathbb{R}}/(\Lambda_\tau)_{\mathbb{R}} \simeq \mathbb{R}^2$. Then the following *balancing condition* must hold in $Q^{\mathrm{gp}} \otimes \check{\Lambda}_v$:

$$\sum_{i=1}^{k} \kappa_{\rho_i}(\varphi) \otimes n_i = 0. \tag{1.7}$$

The balancing condition assures that locally φ has a single-valued representative, even in higher codimension. In this way the MPA-functions of [GrSi2], [GrSi4] can be interpreted as *tropical divisors* on B.

2) In [GHK1] the monoid Q comes with a monoid homomorphism $\mathrm{NE}(Y) \to Q$ from the cone of classes of effective curves of the rational surface Y. The convex MPA-function is obtained by defining κ_ρ for an edge $\rho \in \mathscr{P}$ to be the class of the component $D_\rho \subseteq D$.

Analogous statements hold in [GHKS] with $\mathrm{NE}(Y)$ replaced by $\mathrm{NE}(\mathcal{Y})$, the cone of effective curve classes in the total space of the degeneration $\mathcal{Y} \to T$, and with $D_\rho \subseteq \mathcal{Y}_0$ the double curve corresponding to $\rho \in \mathscr{P}^{[1]}$.

There is also a universal MPA-function. It takes values in a free monoid and even happens to be convex. Denote by $\underline{\mathrm{MPA}}$ the category of convex MPA-functions on B taking values in arbitrary commutative monoids Q and with morphisms from $\varphi_1 \in \mathrm{MPA}(B, Q_1)$ to $\varphi_2 \in \mathrm{MPA}(B, Q_2)$ the homomorphisms $h : Q_1 \to Q_2$ with $\varphi_2 = h \circ \varphi_1$.

Proposition 1.12. *a) The monoid* $Q_0 := \mathrm{Hom}\big(\mathrm{MPA}(B, \mathbb{N}), \mathbb{N}\big)$ *is canonically isomorphic to* $\mathbb{N}^{\tilde{\mathscr{P}}^{[n-1]}_{\mathrm{int}}}$.

b) The Q_0*-valued MPA-function* φ_0 *taking value the generator* $e_{\underline{\rho}} \in \mathbb{N}^{\tilde{\mathscr{P}}^{[n-1]}_{\mathrm{int}}} = Q_0$ *at* $\underline{\rho} \in \tilde{\mathscr{P}}^{[n-1]}_{\mathrm{int}}$ *is an initial object in the category* $\underline{\mathrm{MPA}}$*. In other words, for any monoid* Q *and any* Q*-valued MPA-function* φ *on* B*, there exists a unique monoid homomorphism* $h : Q_0 \to Q$ *such that*

$$\varphi = h \circ \varphi_0.$$

Proof. The $\mathbb{N}^{\tilde{\mathscr{P}}^{[n-1]}_{\mathrm{int}}}$-valued MPA-function φ_0 fulfills the universal property in (b). In fact, if φ is a Q-valued MPA-function the equation $\varphi = h \circ \varphi_0$ holds if and only if h is defined by $h(e_{\underline{\rho}}) := \kappa_{\underline{\rho}}(\varphi)$ for $\underline{\rho} \in \tilde{\mathscr{P}}_{\mathrm{int}}^{[n-1]}$. In particular, $\mathrm{MPA}(B, \mathbb{N}) = \mathrm{Hom}\left(\mathbb{N}^{\tilde{\mathscr{P}}_{\mathrm{int}}^{[n-1]}}, \mathbb{N}\right)$, which shows (a). □

Example 1.13. Let $M = \mathbb{Z}^n$ be a lattice, $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$, $\Gamma \subseteq M$ a rank n sublattice. Consider the real n-torus $B = M_{\mathbb{R}}/\Gamma$, with affine structure induced by the natural affine structure on $M_{\mathbb{R}}$. A polyhedral decomposition $\mathscr{P}$ of B is induced by a Γ-periodic polyhedral decomposition $\bar{\mathscr{P}}$ of $M_{\mathbb{R}}$. Because there are no singularities one imposes (1.7) and thus restricts to multi-valued piecewise linear functions which are locally single-valued, even around codimension ≥ 2 cells of $\mathscr{P}$. Going to the universal cover of $B = B_0$ implies that such a section φ of $\mathcal{MPA}(B, \mathbb{Z})$ is given up to an affine linear function by a piecewise affine function $\bar{\varphi} : M_{\mathbb{R}} \to \mathbb{R}$ affine linear with integral slope on each cell of $\bar{\mathscr{P}}$ and satisfying a periodicity condition

$$\bar{\varphi}(x + \gamma) = \bar{\varphi}(x) + \alpha_\gamma(x) \quad \forall x \in M_{\mathbb{R}}, \gamma \in \Gamma, \tag{1.8}$$

where α_γ is an integral affine linear function depending on γ. Let P be the monoid of all such functions which are in addition (not necessarily strictly) convex: these are those functions whose kink at each codimension one cell of $\mathscr{P}$ is non-negative. (Note that in this case, the kink only depends on the codimension one cell of $\mathscr{P}$, and not on a cell of the barycentric subdivision of $\mathscr{P}$). Then $P^\times = 0$, as the zero multi-valued piecewise linear function is the only convex function all of whose kinks are invertible, that is, 0.

Let $Q = \mathrm{Hom}(P, \mathbb{N})$. We can then assemble all the piecewise linear functions in P into a single function in $\mathrm{MPA}(B, Q^{\mathrm{gp}})$, defined as a function $\varphi_0 : M_{\mathbb{R}} \to Q^{\mathrm{gp}} \otimes_{\mathbb{Z}} \mathbb{R} = \mathrm{Hom}(P, \mathbb{R})$ given by the formula

$$\varphi_0(x) = (\varphi \mapsto \varphi(x)).$$

The kink of φ_0 along $\rho \in \mathscr{P}^{[n-1]}$ is

$$\kappa_\rho(\varphi_0) = (\varphi \mapsto \kappa_\rho(\varphi)) \in \mathrm{Hom}(P, \mathbb{N}).$$

Note that $\kappa_\rho(\varphi_0) \in Q$, so φ_0 is a convex function.

Here we have fixed a single polyhedral decomposition $\mathscr{P}$. It is possible to consider all polyhedral decompositions arising as the domains of linearity of some convex multi-valued piecewise linear function, producing an analogue of the secondary fan for periodic decompositions: this was explored by Alexeev in [Al].

Given an MPA-function $\varphi \in \mathrm{MPA}(B, Q^{\mathrm{gp}})$ we can construct a new polyhedral pseudomanifold $(\mathbb{B}_\varphi, \mathscr{P}_\varphi)$ of dimension $\dim B + \mathrm{rk}\, Q^{\mathrm{gp}}$, along with a $Q^{\mathrm{gp}}_{\mathbb{R}}$-action and an integral affine map $\pi : \mathbb{B}_\varphi \to B$ making $\mathbb{B}_\varphi$ into a $Q^{\mathrm{gp}}_{\mathbb{R}}$-torsor over B. In fact, $\mathbb{B}_\varphi = B \times Q^{\mathrm{gp}}_{\mathbb{R}}$ as a set, but the affine structure of $\mathbb{B}_\varphi$ is twisted by φ as we will explain shortly. The zero section $B \to B \times \{0\} \subseteq \mathbb{B}_\varphi$ defines a piecewise affine right-inverse to π. The image of this section can be viewed as the graph of φ.

Construction 1.14. *(The $Q^{\mathrm{gp}}_{\mathbb{R}}$-torsor $\mathbb{B}_\varphi \to B$)* Let $(B, \mathscr{P})$ be a polyhedral pseudomanifold, Q a toric monoid and $\varphi \in \mathrm{MPA}(B, Q^{\mathrm{gp}})$. Take $\mathbb{B}_\varphi := B \times Q^{\mathrm{gp}}_{\mathbb{R}}$ with polyhedral decomposition

$$\mathscr{P}_\varphi := \{\tau \times Q^{\mathrm{gp}}_{\mathbb{R}} \,|\, \tau \in \mathscr{P}\}.$$

To define the affine structure along a codimension one cell $\underline{\rho} \times Q^{\mathrm{gp}}_{\mathbb{R}}$, $\underline{\rho} \in \tilde{\mathscr{P}}^{[n-1]}_\varphi$ an interior cell, let $\sigma, \sigma' \in \mathscr{P}_{\max}$ be the cells adjacent to $\underline{\rho}$. Let $\delta : \mathrm{Int}\,\sigma \cup \mathrm{Int}\,\sigma' \cup \mathrm{Int}\,\underline{\rho} \to \mathbb{R}$ be the integral affine map with $\delta(\underline{\rho}) = \{0\}$, $\delta(\sigma') \subseteq \mathbb{R}_{\geq 0}$ and surjective differential $D\delta : \Lambda_\sigma \to \mathbb{Z}$. In other words, δ is the signed integral distance from ρ that is positive on σ'. Then for a chart $f : U \to \mathbb{R}^n$ for B_0 with $U \subseteq \mathrm{Int}\,\sigma \cup \mathrm{Int}\,\sigma' \cup \mathrm{Int}\,\underline{\rho}$, define a chart for $\mathbb{B}_\varphi$ by

$$U \times Q^{\mathrm{gp}}_{\mathbb{R}} \longrightarrow \mathbb{R}^n \times Q^{\mathrm{gp}}_{\mathbb{R}}, \quad (x, q) \longmapsto \begin{cases} (f(x), q), & x \in \sigma \\ (f(x), q + \delta(x) \cdot \kappa_{\underline{\rho}}(\varphi)), & x \in \sigma'. \end{cases} \tag{1.9}$$

Here $\kappa_{\underline{\rho}}(\varphi) \in Q^{\mathrm{gp}}$ is the kink of φ along $\underline{\rho}$ defined in Definition 1.6. The projection $\pi : \mathbb{B}_\varphi \to B$ is integral affine and the translation action of $Q^{\mathrm{gp}}_{\mathbb{R}}$ on the second factor of $\mathbb{B}_\varphi = B \times Q^{\mathrm{gp}}_{\mathbb{R}}$ endows $\mathbb{B}_\varphi$ with the structure of a $Q^{\mathrm{gp}}_{\mathbb{R}}$-torsor over B. We may now interpret φ as the zero section $B \to B \times Q^{\mathrm{gp}}_{\mathbb{R}} = \mathbb{B}_\varphi$ since in an affine chart the composition with the projection $\mathbb{B}_\varphi \to Q^{\mathrm{gp}}_{\mathbb{R}}$ indeed represents φ. Note that by (1.9) the zero section of $\mathbb{B}_\varphi$ is only a piecewise integral affine map.

If φ is convex (Definition 1.10) we can also define the *upper convex hull of* φ as the subset $\mathbb{B}^+_\varphi := B \times Q_{\mathbb{R}} \subseteq \mathbb{B}_\varphi$. Here $Q_{\mathbb{R}} = \mathbb{R}_{\geq 0} \cdot Q \subseteq Q^{\mathrm{gp}}_{\mathbb{R}}$ is the convex cone generated by Q. In this case $\partial \mathbb{B}^+_\varphi \subseteq \mathbb{B}_\varphi$ is the image of φ, viewed as a map $B \to \mathbb{B}_\varphi$, plus the preimage of ∂B under the projection $\mathbb{B}^+_\varphi \to B$.

In the situation of Construction 1.14 there are also two sheaves of monoids on B. Later these will carry the exponents of certain rings of Laurent polynomials that provide the local models of the total space of our degeneration.

Definition 1.15. Let $(B, \mathscr{P})$ be a polyhedral pseudomanifold, Q a toric monoid, $\varphi \in \mathrm{MPA}(B, Q)$ a Q-valued convex MPA-function and $\pi : \mathbb{B}_\varphi \to B$ the $Q^{\mathrm{gp}}_{\mathbb{R}}$-torsor defined in Construction 1.14 with canonical section $\varphi : B \to \mathbb{B}_\varphi$. Define the locally constant sheaf of abelian groups

$$\mathcal{P} := \varphi^* \Lambda_{\mathbb{B}_\varphi}$$

on B_0 with fibres $\mathbb{Z}^n \oplus Q^{\mathrm{gp}}$.

For the second sheaf observe that on the interior of a maximal cell σ, the product decomposition $\mathbb{B}_\varphi = B \times Q^{\mathrm{gp}}_{\mathbb{R}}$ is a local isomorphism of affine manifolds independent of any choices. Hence for any $\sigma \in \mathscr{P}_{\max}$ we have a canonical identification

$$\Gamma(\operatorname{Int}\sigma, \mathcal{P}) = \Lambda_\sigma \times Q^{\mathrm{gp}}. \tag{1.10}$$

Furthermore, if $\sigma \cap \partial B \neq \emptyset$ and $\rho \in \mathscr{P}^{[n-1]}$, $\rho \subseteq \sigma \cap \partial B$, define

$$\Lambda_{\sigma,\rho} \subseteq \Lambda_\sigma$$

as the submonoid of tangent vector fields on σ pointing from ρ into σ. In other words, $\Lambda_{\sigma,\rho} \simeq \Lambda_\rho \times \mathbb{N}$ is the preimage of $\mathbb{N}$ under the homomorphism $\Lambda_\sigma \to \Lambda_\sigma/\Lambda_\rho \simeq \mathbb{Z}$ for an appropriate choice of sign for the isomorphism.

Definition 1.16. Denote by $\mathcal{P}^+ \subseteq \mathcal{P}$ the subsheaf with sections over an open set $U \subseteq B_0$ given by $m \in \mathcal{P}(U)$ with $m|_{\operatorname{Int}\sigma} \in \Lambda_\sigma \times Q$ under the identification (1.10), for any $\sigma \in \mathscr{P}_{\max}$. Moreover, if $\rho \cap U \neq \emptyset$ for $\rho \in \mathscr{P}^{[n-1]}$, $\rho \subseteq \partial B$, we require $m|_{\operatorname{Int}\sigma} \in \Lambda_{\sigma,\rho} \times Q$, for $\sigma \in \mathscr{P}_{\max}$ the unique maximal cell containing ρ.

The affine projection $\pi : \mathbb{B}_\varphi \to B$ induces a homomorphism $\pi_* : \mathcal{P} \to \Lambda$ and hence an exact sequence

$$0 \longrightarrow \underline{Q}^{\mathrm{gp}} \longrightarrow \mathcal{P} \xrightarrow{\pi_*} \Lambda \longrightarrow 0 \tag{1.11}$$

of sheaves on B_0. Note that the action of Q^{gp} on the stalks of $\mathcal{P}$ defined by this sequence is induced by the $Q^{\mathrm{gp}}_{\mathbb{R}}$-action on $\mathbb{B}_\varphi$.

2. Wall structures

Throughout this section we fix a polyhedral pseudomanifold $(B, \mathscr{P})$ as introduced in Construction 1.1 and a Noetherian base ring A. The base ring is completely arbitrary subject to the Noetherian condition unless otherwise stated. Let moreover be given a toric monoid Q and a convex MPA function φ on B with values in Q (Definition 1.10). Let $I \subseteq A[Q]$ be an ideal and write $I_0 := \sqrt{I}$ for the radical ideal of I. For any $\underline{\rho} \in \tilde{\mathscr{P}}^{[n-1]}_{\mathrm{int}}$ we assume $z^{\kappa_{\underline{\rho}}(\varphi)} \in I_0$. As a matter of notation, the monomial in $A[Q]$ associated to $m \in Q$ is denoted z^m. So throughout the paper z has a special meaning as a dummy variable in our monoid rings.

An important special case is that $I \subseteq I_0$ is generated by a monoid ideal in Q, but we do not want to restrict to this case.[4] Note also that we do not assume $Q^\times = \{0\}$.

From this data we are first going to construct a non-normal but reduced scheme X_0 over $\operatorname{Spec}(A[Q]/I_0)$ by gluing together toric varieties along toric divisors.

[4]The meaning of writing the base ring as $A[Q]/I$ is that in this form it comes with a chart $Q \to A[Q]/I$ for a log structure that is implicit in the construction.

Write $X_0^\circ \subseteq X_0$ for the complement of the toric strata of codimension at least two.

In a second step we assume given a *wall structure*. We then produce a deformation $\mathfrak{X}^\circ \to \operatorname{Spec}(A[Q]/I)$ of X_0°.

In the affine and projective cases, a third step, treated only in Sections 3 and 4, extends the deformation over the deleted codimension two locus by constructing enough global sections of an ample line bundle.

Examples 2.1. 1) In the setup of [GrSi2], [GrSi4] we considered the case $Q = \mathbb{N}$ and A some base ring of characteristic 0, usually a field or $\mathbb{Z}$.

2) In the case of [GHK1], as outlined in Example 1.3,2, the monoid Q is taken to be a submonoid of $H_2(Y, \mathbb{Z})$ such that $Q^{\mathrm{gp}} = H_2(Y, \mathbb{Z})$ and $\mathrm{NE}(Y) \subseteq Q$, where $\mathrm{NE}(Y)$ denotes the cone of effective curves. The latter monoid frequently is not finitely generated, so it is usually convenient to choose Q to be a larger but finitely generated monoid. The ideal I_0 is often taken to be the ideal of a closed toric stratum of $\operatorname{Spec} \Bbbk[Q]$, for example the ideal of the smallest toric stratum.

In the case of [GHKS], also outlined in Example 1.3,2, we typically work with a finitely generated monoid Q with $\mathrm{NE}(\mathcal{Y}/T) \subseteq Q \subseteq N_1(\mathcal{Y}/T)$, and again I_0 will be the ideal of a closed toric stratum of $\operatorname{Spec} \Bbbk[Q]$. Here $N_1(\mathcal{Y}/T)$ is the group of numerical equivalence classes of algebraic 1-cycles with integral coefficients.

2.1. **Construction of X_0.** Given a polyhedral pseudomanifold $(B, \mathscr{P})$ and convex MPA function φ with values in the toric monoid Q, we construct here the scheme X_0 along with a projective morphism to an affine scheme W_0. Both X_0 and W_0 are reduced but reducible schemes over $A[Q]/I_0$ whose irreducible components are toric varieties.

The construction is easiest by writing down the respective (homogeneous) coordinate rings. Recall that if $\sigma \subseteq \mathbb{R}^n$ is an integral polyhedron, the *cone over* σ is

$$\mathbf{C}\sigma := \mathrm{cl}\big(\mathbb{R}_{\geq 0} \cdot (\sigma \times \{1\})\big) \subseteq \mathbb{R}^n \times \mathbb{R}. \tag{2.1}$$

The closure is necessary to deal with unbounded polyhedra. In fact, if $\sigma = \sigma_0 + \sigma_\infty$ with σ_0 bounded and σ_∞ a cone, then $\mathbf{C}\sigma$ is the Minkowski sum of $\mathbb{R}_{\geq 0} \cdot (\sigma_0 \times \{1\})$ with $\sigma_\infty \times \{0\}$, and the two subcones only intersect in the origin, the tip of $\mathbf{C}\sigma$. The proof of this statement is straightfoward by writing down the inequalities defining σ. Note also that the cone σ_∞ is uniquely determined by σ; it is called the *asymptotic cone* (or *recession cone*) of σ.

For $d > 0$ an integer, rescaling by $1/d$ defines a bijection

$$\mathbf{C}\sigma \cap \big(\mathbb{Z}^n \times \{d\}\big) \longrightarrow \sigma \cap \frac{1}{d}\mathbb{Z}^n$$

between the integral points of $\mathbf{C}\sigma$ of height d and $1/d$-integral points of σ. As a matter of notation define for $d \geq 0$ (now including 0)

$$B\big(\tfrac{1}{d}\mathbb{Z}\big) := \bigcup_{\sigma\in\mathscr{P}_{\max}} \mathbf{C}\sigma \cap \big(\Lambda_\sigma \times \{d\}\big).$$

Here we identify elements in common faces. Thus for $d > 0$ the set $B\big(\frac{1}{d}\mathbb{Z}\big)$ can be identified with the subset of B of points that in some integral affine chart of a cell can be written with coordinates of denominator d.

For any ring S consider the free S-module

$$S[B] := \bigoplus_{d\in\mathbb{N}} S^{B(\frac{1}{d}\mathbb{Z})}$$

with basis elements z^m, $m \in B\big(\frac{1}{d}\mathbb{Z}\big)$ for some $d \in \mathbb{N}$. We turn $S[B]$ into an S-algebra by defining the multiplication of basis elements $z^m \cdot z^{m'} = 0$ unless there exists $\sigma \in \mathscr{P}$ with $m, m' \in \sigma$, and in this case $z^m \cdot z^{m'} := z^{m+m'}$, the sum taken in the monoid $\mathbf{C}\sigma$. The index d defines a $\mathbb{Z}$-grading on $S[B]$, and the homogeneous part of degree d is denoted $S[B]_d$. Note also that because our polytopes have integral vertices, $S[B]$ is generated in degree one.

Now take $S = A[Q]/I_0$ and define

$$W_0 := \operatorname{Spec}(S[B]_0), \quad X_0 := \operatorname{Proj}\big(S[B]\big). \tag{2.2}$$

By construction X_0 is naturally a projective scheme over W_0. To characterize the irreducible components of W_0 and X_0 consider the primary decomposition $I_0 = \bigcap_{i=1}^r \mathfrak{p}_i$ of I_0 and let $S_i := S/\mathfrak{p}_i$. Since I_0 is reduced the $\mathfrak{p}_i$ are prime ideals, and hence $\operatorname{Spec} S = \bigcup_i \operatorname{Spec} S_i$ is a decomposition into integral subschemes.

For an integral polyhedron σ we have the S_i-algebra $S_i[\sigma_\infty \cap \Lambda_\sigma]$ defined by the asymptotic cone σ_∞ of σ and

$$\mathbb{P}_{S_i}(\sigma) := \operatorname{Proj}\big(S_i[\mathbf{C}\sigma \cap (\Lambda_\sigma \oplus \mathbb{Z})]\big),$$

the projective toric variety over $S_i[\sigma_\infty \cap \Lambda_\sigma]$ defined by $\mathbf{C}\sigma$. For a bounded cell σ the asymptotic cone is trivial and hence $S_i[\sigma_\infty \cap \Lambda_\sigma] = S_i$.

Proposition 2.2. *The schemes W_0 and X_0 are reduced. The irreducible components of X_0 are $\mathbb{P}_{S_i}(\sigma)$ with σ running over the maximal cells of $\mathscr{P}$. The irreducible components of W_0 are $\operatorname{Spec}(S_i[\sigma_\infty \cap \Lambda_\sigma])$ with σ running over a subset of the unbounded maximal cells of $\mathscr{P}$.*

Proof. We first give the proof for X_0. For each $\sigma \in \mathscr{P}_{\max}$ denote by $J_\sigma \subseteq S[B]$ the monomial ideal generated by z^m with $m \notin \mathbf{C}\sigma$. We have canonical isomorphisms

$$S[B]/(J_\sigma + \mathfrak{p}_i) \simeq S_i[B]/J_\sigma \simeq S_i[\mathbf{C}\sigma \cap (\Lambda_\sigma \oplus \mathbb{Z})].$$

The ring on the right-hand side is the homogeneous coordinate ring of $\mathbb{P}_{S_i}(\sigma)$, an integral domain as S_i is one. Hence the $J_\sigma + \mathfrak{p}_i$ are prime ideals. Since $\bigcap_{\sigma\in\mathscr{P}_{\max}} J_\sigma = 0$ and the J_σ are not contained in one another we see that $J_\sigma + \mathfrak{p}_i$ with $\sigma \in \mathscr{P}_{\max}$ and $i = 1, \ldots, r$ are the minimal prime ideals in $S[B]$.

For W_0 the analogs of J_σ defined from the asymptotic cones σ_∞ may not be distinct and one has to pick a minimal subset. Otherwise the proof is completely analogous to the case of X_0. □

Remark 2.3. We note that there is not a precise correspondence between unbounded maximal cells of $\mathscr{P}$ and irreducible components of W_0. For example, if $B = \mathbb{R} \times \mathbb{R}/3\mathbb{Z}$ with a subdivision $\mathscr{P}$ induced by the subdivision of $\mathbb{R}$ into two rays with endpoint the origin and the subdivision of $\mathbb{R}/3\mathbb{Z}$ into three unit intervals, W_0 consists of two copies of $\mathbb{A}^1$ glued at the origin. This occurs because all unbounded cells $\mathbb{R}_{\geq 0} \times [i, i+1]$ have the same asymptotic cone, and hence are responsible for the same irreducible component of W_0.

Furthermore, W_0 need not be equidimensional if B has several ends. For example, the above B can easily be modifed by cutting along $\mathbb{R}_{\geq 0} \times \{0\}$ and then gluing in the two edges of $(\mathbb{R}_{\geq 0})^2$ along the cut. Then W_0 has a one-dimensional and two-dimensional irreducible component.

Remark 2.4. Note also that X_0 and W_0 can be written as base change to $A[Q]/I_0$ of the analogous schemes over $\mathbb{Z}$ defined with $A = \mathbb{Z}$, $Q = \mathbb{N}$ and $I_0 = \mathbb{N} \setminus \{0\}$. The scheme X_0 over $\mathbb{Z}$ has one irreducible component for each maximal cell of $\mathscr{P}$.

Example 2.5. Following up on Examples 1.3,2 and 1.11, if we choose Q so that $Q^\times = 0$ and $I_0 = Q \setminus Q^\times$, then in the case of [GHK1], the corresponding scheme X_0 is the *n-vertex*, a union of coordinate planes in affine n-space as labelled:

$$\mathbb{V}_n = \mathbb{A}^2_{x_1,x_2} \cup \cdots \cup \mathbb{A}^2_{x_{n-1},x_n} \cup \mathbb{A}^2_{x_n,x_1} \subseteq \mathbb{A}^n_{x_1,\ldots,x_n}.$$

Here n is the number of irreducible components of $D \subseteq Y$. In the case of [GHKS], X_0 is a union of copies of $\mathbb{P}^2$.

A coarser way to state Proposition 2.2 is that X_0 is a union of toric varieties over $S = A[Q]/I_0$ labelled by elements $\sigma \in \mathscr{P}_{\max}$,

$$\mathbb{P}_S(\sigma) = \operatorname{Proj}\big(S[\mathbf{C}\sigma \cap (\Lambda_\sigma \oplus \mathbb{Z})]\big).$$

This viewpoint motivates the definition of toric strata of higher codimension.

Definition 2.6. A closed subset T of X_0 is called a *toric stratum (of dimension k)* if there exists $\tau \in \mathscr{P}$ of dimension k such that T is the intersection of $\mathbb{P}_S(\sigma) \subseteq X_0$, the intersection taken over all $\sigma \in \mathscr{P}_{\max}$ containing τ.

Note that if I_0 is not a prime ideal then toric varieties over S are not irreducible and neither are our toric strata.

It is not hard to see that X_0 is seminormal if the base ring is seminormal. Similarly, we have the essential:

Proposition 2.7. *Every fibre of $X_0 \to \operatorname{Spec} S$ satisfies Serre's condition S_2. If S satisfies Serre's condition S_2, so does X_0.*

Proof. The second statement follows from the first by [BH], Prop. 2.1.16,(b). Thus we can consider the case when S is a field. Further, X_0 satisfies Serre's condition S_1 since X_0 is reduced.

Thus given $x \in X_0$ a point of height ≥ 2, we need to show $\mathcal{O}_{X_0,x}$ has depth ≥ 2. There is a minimal toric stratum of X_0 containing x, indexed by $\tau \in \mathscr{P}$. If $v \in \operatorname{Int}(\mathbf{C}\tau) \cap (\Lambda_\tau \oplus \mathbb{Z})$, let $z^v \in S[B]$ be the corresponding monomial. Denoting by $S[B]_{(z^v)}$ the homogeneous degree 0 part of the localization $S[B]_{z^v}$, we obtain $U_v = \operatorname{Spec}(S[B]_{(z^v)})$ is an affine open neighbourhood of x.

This neighbourhood can be described combinatorially as follows. Let $\bar{v} \in \tau$ be the image of v under the projection $\mathbf{C}\tau \setminus (\tau_\infty \times \{0\}) \to \tau$ given by $(v, r) \mapsto v/r$. Necessarily $\bar{v} \in \operatorname{Int} \tau$. We can construct a polyhedral cone complex B_v by gluing together the tangent wedges at $\bar{v}$ to maximal cells of $\mathscr{P}$ containing τ, and then $U_v = \operatorname{Spec}(S[B_v]_0)$. If $\tau \subseteq \sigma \in \mathscr{P}$, then we write σ_v for the tangent wedge to σ at $\bar{v}$, and σ_v is a cell in B_v. Note that B_v retains the same S_2 condition as B, and in particular if $\dim \tau \leq n-2$, then $B_v \setminus \tau_v$ is connected. Further, τ_v is a vector space and $B_v = B'_v \times \tau_v$, corresponding to a decomposition $U_v = \operatorname{Spec}(S[B'_v]_0) \times \mathbb{G}_m^{\dim \tau}$.

If the image of x under the projection $U_v \to \mathbb{G}_m^{\dim \tau}$ is height ≥ 1, then a regular sequence of length two in $\mathcal{O}_{U_v,x}$ is easily constructed. Thus we can assume x projects to the generic point of $\mathbb{G}_m^{\dim \tau}$, and so after replacing S with a field extension, we can assume x is the unique zero-dimensional stratum of $\operatorname{Spec}(S[B'_v]_0)$. If $\dim \tau = n$ or $n-1$, then x is a height zero or one point, and there is nothing to show. Thus we may assume that $\dim B'_v \geq 2$, and if τ now denotes the unique zero-dimensional cell of B'_v, then $B'_v \setminus \tau$ is connected.

The result now follows from [BBR], Theorem 1.1. Indeed, write $\mathscr{P}'_v$ for the polyhedral cone complex on B'_v. This is a poset ordered by inclusion, and carries the order topology. Let $\mathcal{F}$ denote the sheaf of S-algebras on $\mathscr{P}'_v$ whose stalk at $\sigma \in \mathscr{P}'_v$ is the ring $S[\sigma \cap \Lambda_\sigma]$. It follows from the criterion of [Yu], Cor. 1.12 that $\mathcal{F}$ is flasque. Also, $\Gamma(\mathscr{P}'_v, \mathcal{F}) = S[B'_v]$. If I is the ideal of the point x, then the hypotheses of [BBR], Theorem 1.1 are satisfied and the local cohomology $H^1_I(S[B'_v])$ is calculated using the formula of that theorem. This is seen to be zero from the connectedness of $B'_v \setminus \tau$, which implies the desired depth statement. $\square$

2.2. **Monomials, rings and gluing morphisms.** Recall from Construction 1.14 that we interpreted φ as a piecewise affine section of the $Q^{\mathrm{gp}}_{\mathbb{R}}$-torsor $\pi : \mathbb{B}_\varphi \to B$, and recall the sheaves $\mathcal{P} = \varphi^*\Lambda_{\mathbb{B}_\varphi}$ and $\mathcal{P}^+ \subseteq \mathcal{P}$ on B_0 from Definitions 1.15 and 1.16. Denote by $A[\mathcal{P}^+]$ the sheaf of $A[Q]$-algebras on B_0 with stalk at x the monoid ring $A[\mathcal{P}^+_x]$. The sheaf of ideals generated by $I \subseteq A[Q]$ is denoted $\mathcal{I}$.

Definition 2.8. A *monomial* at $x \in B_0$ is a formal expression az^m with $a \in A$ and $m \in \mathcal{P}^+_x$. A monomial az^m at x has *tangent vector* $\overline{m} := \pi_*(m) \in \Lambda_x$ with π_* defined in (1.11). Moreover, for $\sigma \in \mathscr{P}_{\max}$ containing x, the σ*-height* $\mathrm{ht}_\sigma(m) \in Q$ of $m \in \mathcal{P}^+_x$ is the projection of m to the second component under the identification (1.10).

By abuse of notation we also refer to elements $m \in \mathcal{P}^+_x$ as monomials.

Example 2.9. Let $B = \mathbb{R}^n$, $\mathscr{P}$ be the fan defining $\mathbb{P}^n$, with rays generated by the standard basis vectors $e_1, \ldots, e_n$ and $e_0 := -e_1 - \cdots - e_n$. Let $Q = \mathbb{N}$, and take $\varphi : B \to Q^{\mathrm{gp}}_{\mathbb{R}} = \mathbb{R}$ to be the piecewise linear function taking the value 0 at $0, e_1, \ldots, e_n$ and the value 1 at e_0. Then $\mathcal{P}$ is the constant sheaf with stalks $\mathbb{Z}^n \times \mathbb{Z}$. The stalk $\mathcal{P}^+_0$ of $\mathcal{P}^+$ at 0 is the monoid $\{(m, r) \mid m \in \mathbb{Z}^n, r \geq \varphi(m)\} \subseteq \mathbb{Z}^{n+1}$. Note this monoid is isomorphic to $\mathbb{N}^{n+1}$, generated by $(e_1, 0), \ldots, (e_n, 0), (e_0, 1)$. For general $x \in \mathbb{R}^n$, if x lies in the interior of the cone generated by $\{e_i \mid i \in I\}$, then $\mathcal{P}^+_x$ is the localization of $\mathcal{P}^+_0$ at the elements $\{(e_i, \varphi(e_i)) \mid i \in I\}$. This localization is abstractly isomorphic to $\mathbb{Z}^{\#I} \times \mathbb{N}^{n+1-\#I}$. Note that $\mathrm{Spec}\, \Bbbk[\mathcal{P}^+_0] \to \mathrm{Spec}\, \Bbbk[Q]$ induced by the obvious inclusion $Q \to \mathcal{P}^+_0$ is a reduced normal crossings degeneration of an algebraic torus to a union of affine spaces.

The aim of this section is to construct a flat $A[Q]/I$-scheme $\mathfrak{X}^\circ$ by gluing spectra of $A[Q]/I$-algebras that are quotients of $A[\mathcal{P}^+_x]$ for $x \in B_0$. Note first that for $\tau \in \mathscr{P}$ parallel transport inside $\tau \setminus \Delta$ induces canonical identifications $\mathcal{P}_x = \mathcal{P}_y$ for x, y in the same connected component of $\tau \setminus \Delta$. This identification maps $\mathcal{P}^+_x$ to $\mathcal{P}^+_y$ and $\mathcal{I}_x$ to $\mathcal{I}_y$ and hence induces an identification of rings

$$A[\mathcal{P}^+_x] \longrightarrow A[\mathcal{P}^+_y] \tag{2.3}$$

mapping $\mathcal{I}_x$ to $\mathcal{I}_y$. There are thus only finitely many rings to be considered, one for each $\sigma \in \mathscr{P}_{\max}$ and one for each $\underline{\rho} \in \tilde{\mathscr{P}}^{[n-1]}_{\mathrm{int}}$ an $(n-1)$-cell of the barycentric subdivision of some $\rho \in \mathscr{P}^{[n-1]}_{\mathrm{int}}$. In the case of a maximal cell σ with $\dim(\sigma \cap \partial B) = n - 1$ there is in addition one more ring for each $\rho \in \mathscr{P}^{[n-1]}$ with $\rho \subseteq \sigma \cap \partial B$.

For $\sigma \in \mathscr{P}_{\max}$ choose $x \in \mathrm{Int}\, \sigma$ and define

$$R_\sigma := A[\mathcal{P}^+_x]/\mathcal{I}_x. \tag{2.4}$$

In view of (2.3) the $A[Q]/I$-algebra R_σ is defined uniquely up to unique isomorphism. Moreover, by (1.10) there is a canonical isomorphism

$$R_\sigma = (A[Q]/I)[\Lambda_\sigma]. \tag{2.5}$$

In particular, $\mathrm{Spec}(R_\sigma)$ is an algebraic torus over $\mathrm{Spec}(A[Q]/I)$ of dimension $n = \mathrm{rk}\,\Lambda_\sigma$.

Similarly, if $\rho \in \mathscr{P}^{[n-1]}$ is a non-interior codimension one cell with adjacent maximal cell σ choose $x \in \mathrm{Int}\,\rho$ and define

$$R_{\sigma,\rho} := A[\mathcal{P}_x^+]/\mathcal{I}_x = (A[Q]/I)[\Lambda_{\sigma,\rho}]. \tag{2.6}$$

The canonical inclusion

$$R_{\sigma,\rho} \longrightarrow R_\sigma \tag{2.7}$$

exhibits R_σ as the localization of $R_{\sigma,\rho}$ by the monomial associated to the (unique) toric divisor of $\mathrm{Spec}\,R_{\sigma,\rho}$.

For an interior codimension one cell $\underline{\rho} \in \tilde{\mathscr{P}}^{[n-1]}$ the situation is a little more subtle. If $x \in \underline{\rho}$, $y \in \underline{\rho}'$ are contained in the same $\rho \in \mathscr{P}^{[n-1]}$ then parallel transport inside an adjacent maximal cell σ still induces an isomorphism $A[\mathcal{P}_x^+]/\mathcal{I}_x \to A[\mathcal{P}_y^+]/\mathcal{I}_y$; but if the affine structure does not extend over $\mathrm{Int}\,\rho$ the isomorphism depends on the choice of σ and hence is not canonical. The naive gluing would thus not fulfill the cocycle condition even locally. To cure this problem we now adjust $A[\mathcal{P}_x^+]/\mathcal{I}_x$ to arrive at the correct rings $R_{\underline{\rho}}$.

For $x \in \rho \setminus \Delta$, $\rho \in \mathscr{P}^{[n-1]}$, there is a submonoid $\Lambda_\rho \times Q^{\mathrm{gp}} \subseteq \mathcal{P}_x$. Under the identification $\mathcal{P}_x = \Lambda_{\mathbb{B}_\varphi,\varphi(x)}$ (Definition 1.15) this submonoid equals $\Lambda_{\rho\times Q^{\mathrm{gp}}_{\mathbb{R}}}$, the integral tangent space of the cell $\rho\times Q^{\mathrm{gp}}_{\mathbb{R}} \subseteq \mathbb{B}_\varphi$. This submonoid is invariant under parallel transport in a neighbourhood of $\mathrm{Int}\,\rho$. In particular, for any $x \in \rho \setminus \Delta$ we obtain a subring $A[\Lambda_\rho \times Q] \subseteq A[\mathcal{P}_x^+]$ and similarly modulo $\mathcal{I}_x$. To generate $A[\mathcal{P}_x^+]$ as an $A[\Lambda_\rho \times Q]$-algebra let $\xi \in \Lambda_x$ generate $\Lambda_x/\Lambda_\rho \simeq \mathbb{Z}$. Then there are unique lifts $Z_+, Z_- \in \mathcal{P}_x$ of $\pm\xi$ with

$$A[\mathcal{P}_x^+]/\mathcal{I}_x \simeq (A[Q]/I)[\Lambda_\rho][Z_+, Z_-]/(Z_+Z_- - z^{\kappa_{\underline{\rho}}}). \tag{2.8}$$

Here $\kappa_{\underline{\rho}} = \kappa_{\underline{\rho}}(\varphi) \in Q$ is the kink of φ along the $(n-1)$-cell $\underline{\rho} \in \tilde{\mathscr{P}}^{[n-1]}_{\mathrm{int}}$ of the barycentric subdivision containing x. Indeed, if $\varphi_x : \Lambda_x \to Q^{\mathrm{gp}}$ is a local representative of φ at x with $\varphi_x(x) = 0$ then

$$\mathcal{P}_x^+ = \big\{(m,q) \in \Lambda_x \times Q^{\mathrm{gp}} \,\big|\, q \in \varphi(m) + Q\big\},$$

and $Z_+ = z^{(\xi,\varphi(\xi))}$, $Z_- = z^{(-\xi,\varphi(-\xi))}$, $Z_+Z_- = z^{(0,\kappa_{\underline{\rho}})}$. Note that changing ξ to $\xi + m$ with $m \in \Lambda_\rho$ changes Z_+ to $z^{(m,\varphi(m))}\cdot Z_+$ and Z_- to $z^{(-m,-\varphi(m))}\cdot Z_-$. In particular, the isomorphism of (2.8) implicitly depends on the choice of ξ. For each ρ we therefore choose an adjacent maximal cell $\sigma = \sigma(\rho)$ and a tangent vector $\xi = \xi(\rho) \in \Lambda_\sigma$ with

$$\Lambda_\rho + \mathbb{Z} \cdot \xi(\rho) = \Lambda_\sigma. \tag{2.9}$$

To fix signs we require that $\xi(\rho)$ points from ρ into $\sigma(\rho)$.

We now assume that for each $\underline{\rho} \in \tilde{\mathscr{P}}^{[n-1]}_{\rm int}$ we have a polynomial $f_{\underline{\rho}} \in (A[Q]/I)[\Lambda_\rho]$ with the compatibility property that if $\underline{\rho}, \underline{\rho}' \subseteq \rho$ then

$$z^{\kappa_{\underline{\rho}'}} f_{\underline{\rho}'} = z^{m_{\underline{\rho}'\underline{\rho}}} \cdot z^{\kappa_{\underline{\rho}}} f_{\underline{\rho}}. \tag{2.10}$$

Now define

$$R_{\underline{\rho}} := (A[Q]/I)[\Lambda_\rho][Z_+, Z_-]/(Z_+Z_- - f_{\underline{\rho}} \cdot z^{\kappa_{\underline{\rho}}}). \tag{2.11}$$

As an abstract ring this is independent of all choices, but the interpretation of $Z_\pm$ depends on the above choice of a maximal cell $\sigma(\rho) \supset \rho$ and $\xi = \xi(\rho) \in \Lambda_\sigma$. This choice will become important in the gluing to the rings R_σ, $\sigma \in \mathscr{P}_{\max}$, see (2.14) below.

The rings $R_{\underline{\rho}}$ are now compatible with local parallel transport. Specifically, let $\rho \in \mathscr{P}^{[n-1]}$ contain $\underline{\rho}, \underline{\rho}' \in \tilde{\mathscr{P}}^{[n-1]}_{\rm int}$. Let $Z_\pm \in R_{\underline{\rho}}$ be the lifts of $\pm\xi(\rho)$ as defined above, and $Z'_\pm \in R_{\underline{\rho}'}$ the lifts for $\underline{\rho}'$. Since $\xi(\rho)$ is a vector field on $\sigma = \sigma(\rho)$, parallel transport of monomials inside σ maps Z_+ to Z'_+. Use parallel transport inside the other maximal cell $\sigma' \supset \rho$ to define the image of Z_- in $R_{\underline{\rho}'}$. Let $y \in \operatorname{Int} \underline{\rho}'$. The result ξ' of parallel transport of $-\xi(\rho) \in \Lambda_x$ through σ' differs from $-\xi(\rho) \in \Lambda_y$ by monodromy around a loop passing from $y \in \operatorname{Int} \underline{\rho}'$ via σ to $x \in \operatorname{Int} \underline{\rho}$ and back to y via σ'. By (1.3) we obtain

$$\xi' = -\xi(\rho) + \check{d}_\rho(-\xi(\rho)) \cdot m_{\underline{\rho}'\underline{\rho}} = -\xi(\rho) - m_{\underline{\rho}'\underline{\rho}} = -\xi(\rho) + m_{\underline{\rho}\underline{\rho}'}. \tag{2.12}$$

This computation suggests that we identify $R_{\underline{\rho}}$ and $R_{\underline{\rho}'}$ by mapping Z_+ to Z'_+ and Z_- to $z^{m_{\underline{\rho}\underline{\rho}'}} Z'_-$.

Lemma 2.10. *Let $\rho \in \mathscr{P}^{[n-1]}$, $\rho \not\subseteq \partial B$, contain $\underline{\rho}$, $\underline{\rho}' \in \tilde{\mathscr{P}}^{[n-1]}_{\rm int}$ and let $Z_\pm \in R_{\underline{\rho}}$, $Z'_\pm \in R_{\underline{\rho}'}$ be lifts of $\pm\xi(\rho)$ as defined above. Then there is a canonical isomorphism of $(A[Q]/I)[\Lambda_\rho]$-algebras*

$$R_{\underline{\rho}} \longrightarrow R_{\underline{\rho}'} \tag{2.13}$$

mapping Z_+ to Z'_+ and Z_- to $z^{m_{\underline{\rho}\underline{\rho}'}} Z'_-$.

Proof. By (2.10) we have the equality $z^{\kappa_{\underline{\rho}}} f_{\underline{\rho}} = z^{m_{\underline{\rho}\underline{\rho}'}} \cdot z^{\kappa_{\underline{\rho}'}} f_{\underline{\rho}'}$ in $(A[Q]/I)[\Lambda_\rho]$. Thus under the stated map the relation $Z_+Z_- - f_{\underline{\rho}} z^{\kappa_{\underline{\rho}}}$ in $R_{\underline{\rho}}$ maps to

$$Z'_+ z^{m_{\underline{\rho}\underline{\rho}'}} Z'_- - z^{\kappa_{\underline{\rho}}} f_{\underline{\rho}} = z^{m_{\underline{\rho}\underline{\rho}'}} \big(Z'_+ Z'_- - z^{\kappa_{\underline{\rho}'}} f_{\underline{\rho}'}\big).$$

From this computation the statement is immediate. $\square$

If $\sigma \in \mathscr{P}_{\max}$ contains $\underline{\rho} \in \tilde{\mathscr{P}}^{[n-1]}_{\rm int}$ there is also a canonical localization homomorphism

$$R_{\underline{\rho}} \longrightarrow \begin{cases} (R_{\underline{\rho}})_{Z_+} = R_\sigma & , \ \sigma = \sigma(\rho) \\ (R_{\underline{\rho}})_{Z_-} = R_\sigma & , \ \sigma \neq \sigma(\rho). \end{cases} \tag{2.14}$$

The isomorphism $(R_{\underline{\rho}})_{Z_+} = R_\sigma$ is defined by eliminating Z_- via the equation $Z_+Z_- = f_{\underline{\rho}} \cdot z^{\kappa_{\underline{\rho}}}$ and mapping Z_+ to $z^{\xi(\rho)}$. The other monomials are identified via $\Lambda_\rho \subseteq \Lambda_\sigma$. Note that this map is not injective since Z_-^l maps to zero for $l \gg 0$, due to the fact that $\kappa_{\underline{\rho}} \in I_0$ and $I_0 = \sqrt{I}$. A similar reasoning holds for $(R_{\underline{\rho}})_{Z_-}$, using parallel transport through $\underline{\rho}$ to view $\xi(\rho) \in \Lambda_{\sigma(\rho)}$ as an element of Λ_σ.

2.3. **Walls and consistency.** The rings R_σ, $R_{\sigma,\rho}$ and $R_{\underline{\rho}}$ together with the isomorphisms (2.13) and the localization homomorphisms (2.14), (2.7) form a category (or inverse system) of $A[Q]/I$-algebras. Choosing for each $\rho \in \mathscr{P}^{[n-1]}$ one $\underline{\rho} \in \tilde{\mathscr{P}}^{[n-1]}_{\rm int}$ with $\underline{\rho} \subseteq \rho$ defines an equivalent subcategory. Taking Spec of this subcategory then defines a direct system of affine schemes and open embeddings with the property that the only non-trivial triple fibre products come from maximal cells σ with $\sigma \cap \partial B \neq \emptyset$ and the codimension one cells $\rho \subseteq \partial B \cap \sigma$. Fixing σ, this latter system of schemes has a limit, the open subscheme $\bigcup_{\rho \subseteq \sigma\cap\partial B} \operatorname{Spec}\big(R_{\sigma,\rho}\big)$ of the toric variety $\mathbb{P}_\sigma$ with momentum polytope σ. Let $D_{\rm int} \subseteq \mathbb{P}_\sigma$ be the union of toric divisors corresponding to facets $\rho \subseteq \sigma$ with $\rho \not\subseteq \partial B$. Then the complement of this open subscheme in $\mathbb{P}_\sigma \setminus D_{\rm int}$ is the union of toric strata of codimension larger than 1. Hence there exists a colimit of our category of schemes as a separated scheme over $\operatorname{Spec}(A[Q]/I)$. It has an open cover by the affine schemes $\operatorname{Spec} R_\sigma$, $\operatorname{Spec} R_{\sigma,\rho}$ and $\operatorname{Spec} R_{\underline{\rho}}$, for the chosen subset of $\underline{\rho}$'s in $\tilde{\mathscr{P}}^{[n-1]}_{\rm int}$.

This scheme is not quite what we want, since it is both a bit too simple and it may not possess enough regular functions semi-locally.[5] Rather we will introduce higher order corrections to the functions $f_{\underline{\rho}}$ and to the gluing morphisms. The latter are carried by locally polyhedral subsets of B of codimension one, called walls. For a polyhedral subset $\mathfrak{p}$ of some $\sigma \in \mathscr{P}_{\max}$ with $\Delta \cap \operatorname{Int}\mathfrak{p} = \emptyset$ we write $\Lambda_{\mathfrak{p}} \subseteq \Lambda_\sigma$ for the vectors tangent to $\mathfrak{p}$.

Definition 2.11. 1) A *wall* on our polyhedral pseudomanifold $(B, \mathscr{P})$ is a codimension one rational polyhedral subset $\mathfrak{p} \not\subseteq \partial B$ of some $\sigma \in \mathscr{P}_{\max}$ with $\operatorname{Int}\mathfrak{p} \cap \Delta = \emptyset$, along with an element

$$f_{\mathfrak{p}} = \sum_{m \in \mathcal{P}_x^+,\, \overline{m} \in \Lambda_{\mathfrak{p}}} c_m z^m \in A[\mathcal{P}_x^+],$$

[5] On a technical level, consistency in codimension two (Definition 3.9) may fail for this uncorrected scheme.

for $x \in \operatorname{Int}\mathfrak{p}$. Identifying $\mathcal{P}_y$ with $\mathcal{P}_x$ by parallel transport inside σ we require $m \in \mathcal{P}_y^+$ for any $y \in \mathfrak{p} \setminus \Delta$ when $c_m \neq 0$. Moreover, the following holds:

<u>codim = 0:</u> If $\mathfrak{p} \cap \operatorname{Int}\sigma \neq \emptyset$ then $f_\mathfrak{p} \equiv 1$ modulo I_0.
<u>codim = 1:</u> If $\mathfrak{p} \subseteq \underline{\rho}$ for some $\underline{\rho} \in \tilde{\mathscr{P}}_{\text{int}}^{[n-1]}$ then $f_\mathfrak{p} \equiv f_{\underline{\rho}}$ modulo I_0.

Here codim refers to the *codimension of* $\mathfrak{p}$, defined as the codimension of the minimal cell of $\mathscr{P}$ containing $\mathfrak{p}$. Codimension one walls are also called *slabs*, denoted $\mathfrak{b}$.

2) A *wall structure* on $(B, \mathscr{P})$ is a set $\mathscr{S}$ of walls with the following properties:

(a) Any cell $\sigma \in \mathscr{P}_{\max}$ contains only finitely many walls; the underlying polyhedral sets of $\mathscr{S}$ are the maximal cells of a polyhedral decomposition $\mathscr{P}_\mathscr{S}$ of a closed subset of B of dimension $n-1$. In particular, the interior of a wall does not intersect any other wall.

(b) $\displaystyle\bigcup_{\rho \in \mathscr{P}_{\text{int}}^{[n-1]}} \rho \subseteq |\mathscr{S}|$.

(c) The closure $\mathfrak{u}$ of any connected component of $B \setminus |\mathscr{S}|$ is a convex polyhedron. Moreover, there is at most one $\rho \in \mathscr{P}^{[n-1]}$ with $\rho \subseteq \partial B$ and $\dim(\mathfrak{u} \cap \rho) = n-1$.

A polyhedron $\mathfrak{u}$ as in (c) is called a *chamber* of the wall structure. Two chambers $\mathfrak{u}, \mathfrak{u}'$ are *adjacent* if $\dim \mathfrak{u} \cap \mathfrak{u}' = n-1$. A chamber $\mathfrak{u}$ with $\dim(\mathfrak{u} \cap \partial B) = n-1$ is called a *boundary chamber*, otherwise an *interior chamber*. Elements $\mathfrak{j} \in \mathscr{P}_\mathscr{S}$ of codimension two are called *joints*. A joint $\mathfrak{j}$ with $\mathfrak{j} \subseteq \partial B$ is called a *boundary joint*, otherwise an *interior joint*. The *codimension* $k \in \{0,1,2\}$ of a joint is the codimension of the smallest cell of $\mathscr{P}$ containing $\mathfrak{j}$.

Remark 2.12. 1) In the definition of wall structure we do not require that the walls and chambers form a polyhedral decomposition of B. A typical phenomenon is that a wall $\mathfrak{p} \subseteq \sigma$, $\sigma \in \mathscr{P}_{\max}$, intersects the interior of $\rho \in \mathscr{P}_{\text{int}}^{[n-1]}$, but $\rho \cap \mathfrak{p}$ is not contained in a union of walls on the other adjacent maximal cell $\sigma' \neq \sigma$, $\rho = \sigma \cap \sigma'$.

2) By the definition of walls and the covering condition 2,(b) the discriminant locus Δ is covered by joints. In particular, if $\mathfrak{u}$, $\mathfrak{u}'$ are adjacent chambers then $\operatorname{Int}(\mathfrak{u} \cap \mathfrak{u}') \cap \Delta = \emptyset$. This is different from the convention in [GrSi4] where Δ was chosen transcendental and transverse to all joints. In particular, a slab $\mathfrak{b}$ in [GrSi4] could be disconnected by Δ. See Appendix A.1 for how the construction of [GrSi4] also produces wall structures with the present conventions.

Furthermore, in [GrSi4] walls could intersect in a subset of dimension $n-1$, which in the present definition is excluded by (a). This is, however, no restriction for one can always first subdivide walls to make this only happen if the underlying polyhedral subsets of two walls $\mathfrak{p}', \mathfrak{p}''$ agree. Then in a second step, replace all walls $\mathfrak{p}_i$ with the same underlying polyhedral set by the wall $\mathfrak{p} := \mathfrak{p}_i$ for any i and define $f_\mathfrak{p} := \prod_i f_{\mathfrak{p}_i}$. This process is compatible with taking the composition of the automorphisms associated to walls to be defined in (2.19).

3) By definition any chamber $\mathfrak{u}$ is contained in a unique maximal cell $\sigma = \sigma_\mathfrak{u}$. Thus chambers $\mathfrak{u}, \mathfrak{u}'$ can be adjacent in two ways. (I) If $\sigma_\mathfrak{u} = \sigma_{\mathfrak{u}'}$ then $\mathfrak{u} \cap \mathfrak{u}'$ must be an $(n-1)$-cell of $\mathscr{P}_\mathscr{S}$ intersecting the interior of a maximal cell, hence the underlying set of a codimension zero wall. Otherwise, (II), $\sigma_\mathfrak{u} \cap \sigma_{\mathfrak{u}'}$ has dimension smaller than n, but contains the $(n-1)$-dimensional subset $\mathfrak{u} \cap \mathfrak{u}'$. Hence $\rho = \sigma_\mathfrak{u} \cap \sigma_{\mathfrak{u}'} \in \mathscr{P}^{[n-1]}_{\mathrm{int}}$, and $\mathfrak{u} \cap \mathfrak{u}'$ is covered by the underlying sets of slabs.

4) Condition (c) on boundary chambers is purely technical and can always be achieved by introducing some walls $\mathfrak{p}$ with $f_\mathfrak{p} = 1$.

5) In this paper the $f_{\underline{\rho}}$ are redundant information once we assume a wall structure $\mathscr{S}$ to be given. Moreover, condition (2.10) follows from consistency of $\mathscr{S}$ in codimension one introduced in Definition 2.14 below. On the other hand, the reduction of $f_{\underline{\rho}}$ modulo I_0 determines (and is indeed equivalent to) the log structure induced by the degeneration on the central fibre, see [GrSi2]. Thus these functions already contain crucial information. In the case with locally rigid singularities treated in [GrSi4] the whole wall structure can even be constructed inductively just from this knowledge.

Let now be given a wall structure $\mathscr{S}$ on $(B, \mathscr{P})$. There are three kinds of rings associated to $\mathscr{S}$. First, for any chamber $\mathfrak{u}$ define

$$\boxed{R_\mathfrak{u} := R_\sigma = (A[Q]/I)[\Lambda_\sigma].} \tag{2.15}$$

for the unique $\sigma \in \mathscr{P}_{\max}$ containing $\mathfrak{u}$. Second, if $\mathfrak{u}$ is a boundary chamber, then according to Definition 2.11,2(c) there is a unique $\rho \in \mathscr{P}^{[n-1]}$ with $\dim(\rho \cap \mathfrak{u} \cap \partial B) = n-1$. We then have the subring

$$\boxed{R^\partial_\mathfrak{u} := R_{\sigma,\rho} \subseteq R_\mathfrak{u}.} \tag{2.16}$$

Thus $R_\mathfrak{u}$ and $R^\partial_\mathfrak{u}$ are just different notations for the rings already introduced in (2.4) and (2.6). The third kind of ring is a deformation of the ring $R_{\underline{\rho}}$ from (2.11) given by a slab $\mathfrak{b} \subseteq \underline{\rho}$:

$$\boxed{R_\mathfrak{b} := (A[Q]/I)[\Lambda_\rho][Z_+, Z_-]/(Z_+Z_- - f_\mathfrak{b} \cdot z^{\kappa_{\underline{\rho}}}).} \tag{2.17}$$

We indeed have $R_\mathfrak{b}/I_0 = R_{\underline{\rho}}$ since $f_\mathfrak{b} \equiv f_{\underline{\rho}}$ modulo I_0 according to Definition 2.11,1.

Between the rings $R_{\mathfrak{u}}$, $R^{\partial}_{\mathfrak{u}}$ and $R_{\mathfrak{b}}$ there are two types of localization homomorphisms, namely

$$\boxed{\chi_{\mathfrak{b},\mathfrak{u}} : R_{\mathfrak{b}} \longrightarrow R_{\mathfrak{u}}, \quad \chi^{\partial}_{\mathfrak{u}} : R^{\partial}_{\mathfrak{u}} \longrightarrow R_{\mathfrak{u}}} \tag{2.18}$$

defined as in (2.14) for $\mathfrak{b} \subseteq \mathfrak{u}$ and in (2.7) for $\mathfrak{u}$ a boundary chamber, respectively.

Furthermore, to a codimension zero wall $\mathfrak{p}$ separating interior chambers $\mathfrak{u}$, $\mathfrak{u}'$ (contained in $\sigma \in \mathscr{P}_{\max}$, say) we associate an isomorphism $\theta_{\mathfrak{p}} : R_{\mathfrak{u}} \to R_{\mathfrak{u}'}$ as follows. Let $n_{\mathfrak{p}}$ be a generator of $\Lambda^{\perp}_{\mathfrak{p}} \subseteq \check{\Lambda}_x$ for some $x \in \operatorname{Int} \mathfrak{p}$. Denote by $\mathfrak{u}$, $\mathfrak{u}'$ the two chambers separated by $\mathfrak{p}$ with $n_{\mathfrak{p}} \geq 0$ as a function on $\mathfrak{u}$ in an affine chart mapping x to the origin. Then define

$$\boxed{\theta_{\mathfrak{p}} : R_{\mathfrak{u}} \longrightarrow R_{\mathfrak{u}'}, \quad z^m \longmapsto f_{\mathfrak{p}}^{\langle n_{\mathfrak{p}}, \overline{m}\rangle} z^m.} \tag{2.19}$$

Here we view $f_{\mathfrak{p}}$ as an element of $R^{\times}_{\sigma} = R^{\times}_{\mathfrak{u}'}$ by reduction modulo I. We refer to $\theta_{\mathfrak{p}}$ as the automorphism associated to *crossing the wall* $\mathfrak{p}$ or to *passing from* $\mathfrak{u}$ *to the adjacent chamber* $\mathfrak{u}'$.

If $\dim \mathfrak{p} \cap \partial B = n-2$ then $\mathfrak{p}$ separates two boundary chambers $\mathfrak{u}, \mathfrak{u}'$. Assuming $\mathfrak{p}$ intersects also the interior of some $\rho \in \mathscr{P}^{[n-1]}_{\partial}$ the rings $R^{\partial}_{\mathfrak{u}}$ and $R^{\partial}_{\mathfrak{u}'}$ are the same localization $R_{\sigma,\rho}$ of R_{σ}. In this case there is an induced isomorphism

$$\boxed{\theta^{\partial}_{\mathfrak{p}} : R^{\partial}_{\mathfrak{u}} \to R^{\partial}_{\mathfrak{u}'}.} \tag{2.20}$$

In fact, the requirement of the monomials occurring in $f_{\mathfrak{p}}$ to lie in $\mathcal{P}^{+}_{y}$ for all $y \in \mathfrak{p} \setminus \Delta$ implies that they do not point outward from ∂B. Thus $f_{\mathfrak{p}}$ makes sense as an element of $R^{\partial}_{\mathfrak{u}'}$. This shows $\theta_{\mathfrak{p}}(R^{\partial}_{\mathfrak{u}}) \subseteq R^{\partial}_{\mathfrak{u}'}$. The converse inclusion follows from considering $\theta^{-1}_{\mathfrak{p}}$.

Next we would like to glue the affine schemes $\operatorname{Spec} R_{\mathfrak{b}}$, $\operatorname{Spec} R_{\mathfrak{u}}$ and $\operatorname{Spec} R^{\partial}_{\mathfrak{u}}$ via the natural localization morphisms (2.18), analogous to the discussion in the introductory paragraph of this subsection, but observing the wall crossing isomorphisms (2.19),(2.20) between the $R_{\mathfrak{u}}$. The scheme $\mathfrak{X}^{\circ}$ is thus constructed as the colimit of a category with morphisms generated by all possible wall crossings and the localization homomorphisms. Since now we have many triple intersections we need a compatibility condition for this colimit to be meaningful. Eventually there will be three consistency conditions: (1) Around codimension zero joints. (2) Around codimension one joints. (3) Local consistency in higher codimension tested by broken lines (see Section 3). The last point is only necessary for the construction of enough functions and does not concern us for the moment.

As for consistency around a codimension zero joint $\mathfrak{j}$ let $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ be the walls containing $\mathfrak{j}$. Working in the quotient space $\Lambda_{\sigma,\mathbb{R}}/\Lambda_{\mathfrak{j},\mathbb{R}} \simeq \mathbb{R}^2$, any quotient $\mathfrak{p}_i/\Lambda_{\mathfrak{j},\mathbb{R}}$ is a line segment emanating from the origin. Note that since the $\mathfrak{p}_i$ are maximal cells of the polyhedral decomposition $\mathscr{P}_{\mathscr{S}}$ the line segments intersect pairwise only at the origin. We may assume the $\mathfrak{p}_i$ are labelled in such a way that these line segments are ordered cyclically. Define $\theta_{\mathfrak{p}_i}$ by (2.19) with signs fixed by crossing the walls in a cyclic order.

Definition 2.13. The set of walls $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ containing the codimension zero joint $\mathfrak{j}$ is called *consistent* if

$$\theta_{\mathfrak{p}_r} \circ \ldots \circ \theta_{\mathfrak{p}_1} = \mathrm{id},$$

as an automorphism of R_σ, for $\sigma \in \mathscr{P}_{\max}$ the unique maximal cell containing $\mathfrak{j}$.

A wall structure $\mathscr{S}$ on the polyhedral pseudomanifold $(B, \mathscr{P})$ is *consistent in codimension zero* if for any codimension zero joint $\mathfrak{j}$ the set $\{\mathfrak{p} \in \mathscr{S} \,|\, \mathfrak{p} \subseteq \mathfrak{j}\}$ of walls containing $\mathfrak{j}$ is consistent.

Consistency around a codimension one joint $\mathfrak{j}$ is a little more subtle. There is no condition for a codimension one joint contained in ∂B. Otherwise, let $\rho \in \mathscr{P}$ be the codimension one cell containing $\mathfrak{j}$ and let σ, σ' be the unique maximal cells containing ρ. By the polyhedral decomposition property of $\mathscr{S}$ and since $\mathfrak{j} \not\subseteq \partial B$ there are unique slabs $\mathfrak{b}_1, \mathfrak{b}_2 \subseteq \rho$ with $\mathfrak{j} = \mathfrak{b}_1 \cap \mathfrak{b}_2$. Denote by $\mathfrak{p}_1, \ldots, \mathfrak{p}_r \subseteq \sigma$ and $\mathfrak{p}'_1, \ldots, \mathfrak{p}'_s \subseteq \sigma'$ the codimension zero walls containing $\mathfrak{j}$. We assume that the sequence $\mathfrak{b}_1, \mathfrak{p}_1, \ldots, \mathfrak{p}_r, \mathfrak{b}_2, \mathfrak{p}'_1, \ldots, \mathfrak{p}'_s$ is a cyclic ordering around $\mathfrak{j}$ similarly to the case of codimension one walls.[6] There are then (non-injective) localization homomorphisms

$$\chi_{\mathfrak{b}_i,\sigma} : R_{\mathfrak{b}_i} \longrightarrow R_\sigma, \quad \chi_{\mathfrak{b}_i,\sigma'} : R_{\mathfrak{b}_i} \longrightarrow R_{\sigma'}, \quad i = 1, 2, \tag{2.21}$$

and a composition of wall crossings on either side of ρ:

$$\begin{aligned} \theta &:= \theta_r \circ \theta_{r-1} \circ \ldots \circ \theta_1 : \qquad R_\sigma \to R_\sigma \\ \theta' &:= \theta'_1 \circ \theta'_2 \circ \ldots \circ \theta'_s : \qquad R_{\sigma'} \to R_{\sigma'}. \end{aligned}$$

Now observe that

$$(\chi_{\mathfrak{b}_i,\sigma}, \chi_{\mathfrak{b}_i,\sigma'}) : R_{\mathfrak{b}_i} \longrightarrow R_\sigma \times R_{\sigma'}$$

is injective. In fact, assuming without restriction $\sigma = \sigma(\rho)$ for $\rho = \sigma \cap \sigma'$, we have $\ker(\chi_{\mathfrak{b}_i,\sigma}) \subseteq (Z_-)$, and $\chi_{\mathfrak{b}_i,\sigma'}(Z_-)$ is invertible in $R_{\sigma'}$. The consistency condition is the requirement that $\theta \times \theta'$ induces a well-defined map $R_{\mathfrak{b}_1} \to R_{\mathfrak{b}_2}$.

Definition 2.14. The set $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_r, \mathfrak{p}'_1, \ldots, \mathfrak{p}'_s, \mathfrak{b}_1, \mathfrak{b}_2\}$ of walls and slabs containing the codimension one joint $\mathfrak{j}$ is *consistent* if

$$(\theta \times \theta')\big((\chi_{\mathfrak{b}_1,\sigma}, \chi_{\mathfrak{b}_1,\sigma'})(R_{\mathfrak{b}_1})\big) = (\chi_{\mathfrak{b}_2,\sigma}, \chi_{\mathfrak{b}_2,\sigma'})(R_{\mathfrak{b}_2}).$$

[6] If $\mathfrak{j} \subseteq \Delta$ the quotient space is not well-defined as an affine plane, but only as a union of two affine half-planes, the tangent wedges of σ and σ' along $\sigma \cap \sigma' \in \mathscr{P}^{[n-1]}$. This is enough for our purposes.

In this case we define

$$\theta_{\mathfrak{j}} : R_{\mathfrak{b}_1} \longrightarrow R_{\mathfrak{b}_2} \tag{2.22}$$

as the isomorphism induced by $\theta \times \theta'$.

A wall structure $\mathscr{S}$ on the polyhedral pseudomanifold $(B, \mathscr{P})$ is *consistent in codimension one* if for any codimension one interior joint $\mathfrak{j}$ the set $\{\mathfrak{p} \in \mathscr{S} \,|\, \mathfrak{j} \subseteq \mathfrak{p}\}$ of walls and slabs containing $\mathfrak{j}$ is consistent.

Example 2.15. 1) Wall structures were introduced in [GrSi4], with a slight difference in the treatment of slabs. In [GrSi4], a slab $\mathfrak{b}$ (a codimension 1 wall) could have $(\operatorname{Int} \mathfrak{b}) \cap \Delta \neq \emptyset$. There was not a single function attached to a slab, but rather, one choice of function for each connected component of $\mathfrak{b} \setminus \Delta$, with relations between these functions determined by the local monodromy analogous to (2.10). Indeed, in loc.cit. the discriminant locus was taken with irrational position in such a way that no codimension zero wall could ever contain an open part of Δ. In such a situation consistency in codimension one is equivalent to an equation of the form (2.10) relating the functions on the various connected components of $\mathfrak{b}\setminus\Delta$. In [GrSi4], a wall structure consistent in all codimensions was constructed; in the current setup, we cannot define consistency in codimension two directly but only after the construction of local functions, see §3.2. This wall structure was used to construct a deformation $\mathfrak{X}$ of X_0, rather than just a deformation $\mathfrak{X}^\circ$ of the complement of codimension two strata of X_0 as given in Proposition 2.16 below. The construction of [GrSi4] makes use of local models for the smoothings of X_0 in neighbourhoods of higher codimension strata; here we just use codimension one strata, where the local model is given by (2.17). This makes the construction technically much easier than in [GrSi4].

2) [GHK1] defined the notion of *scattering diagram* on the pair (B, Σ) arising from a pair (Y, D) as in Example 1.3,2. This is a special case of a wall structure on (B, Σ), in which every wall has support a ray with endpoint $0 \in B$. In this case consistency in codimensions zero and one are automatic.

If one is interested in a compact example, with $\bar{B} \subseteq B$ a compact two-dimensional subset as described in Example 1.3,2, a scattering diagram $\mathfrak{D}$ on (B, Σ) gives rise to a wall structure $\mathscr{S}$ on $(\bar{B}, \mathscr{P} = \{\tau \cap \bar{B} \,|\, \tau \in \Sigma\})$. One takes

$$\begin{aligned}\mathscr{S} = \{&(\mathfrak{d} \cap \bar{B}, f_{\mathfrak{d}}) \,|\, (\mathfrak{d}, f_{\mathfrak{d}}) \in \mathfrak{D}, \operatorname{codim} \mathfrak{d} = 0\} \\ &\cup \{(\underline{\rho}, f_{\mathfrak{d}}) \,|\, (\mathfrak{d}, f_{\mathfrak{d}}) \in \mathfrak{D},\ \operatorname{codim} \mathfrak{d} = 1,\ \underline{\rho} \in \tilde{\mathscr{P}}^{[1]},\ \underline{\rho} \subseteq \mathfrak{d}\}.\end{aligned}$$

Note that the only singularity of the affine structure on $\bar{B}$ is at the origin. Thus the barycentric subdivision for the slabs only appears here to conform to the conventions of the present paper. Again, consistency in codimension zero and one is automatic.

2.4. **Construction of $\mathfrak{X}^\circ$.** With the notion of consistency of a wall structure in codimension zero and one at hand (Definitions 2.13 and 2.14) we are now in position to construct our family $\mathfrak{X} \to \operatorname{Spec}(A[Q]/I)$ outside codimension two.

Proposition 2.16. *Let $\mathscr{S}$ be a wall structure on the polyhedral pseudomanifold $(B, \mathscr{P})$. If $\mathscr{S}$ is consistent in codimensions zero and one there exists a unique scheme $\mathfrak{X}^\circ$ flat over $\operatorname{Spec}(A[Q]/I)$ together with open embeddings $\operatorname{Spec} R_{\mathfrak{u}} \to \mathfrak{X}^\circ$, $\operatorname{Spec} R^\partial_{\mathfrak{u}} \to \mathfrak{X}^\circ$ and $\operatorname{Spec} R_{\mathfrak{b}} \to \mathfrak{X}^\circ$ for (boundary) chambers $\mathfrak{u}$ and slabs $\mathfrak{b}$ of $\mathscr{S}$ that are compatible with the morphisms $\theta_{\mathfrak{p}}$ and $\theta^\partial_{\mathfrak{p}}$ for codimension zero walls (2.19),(2.20), $\theta_{\mathfrak{j}}$ for codimension one joints (2.22) and with the open embeddings from (2.18), that is, $\operatorname{Spec} R_{\mathfrak{u}} \to \operatorname{Spec} R_{\mathfrak{b}}$ for $\mathfrak{b} \subseteq \mathfrak{u}$ and with $\operatorname{Spec} R_{\mathfrak{u}} \to \operatorname{Spec} R^\partial_{\mathfrak{u}}$ for $\mathfrak{u}$ a boundary chamber.*

Proof. Define a category $\underline{C}$ whose objects are the following schemes over $A[Q]/I$. We have $U_{\mathfrak{u}} := \operatorname{Spec} R_{\mathfrak{u}}$ for chambers $\mathfrak{u}$ of $\mathscr{S}$, $U^\partial_{\mathfrak{u}} := \operatorname{Spec} R^\partial_{\mathfrak{u}}$ for boundary chambers $\mathfrak{u}$ and $U_{\mathfrak{b}} := \operatorname{Spec} R_{\mathfrak{b}}$ for slabs $\mathfrak{b} \in \mathscr{S}$. The morphisms in $\underline{C}$ are defined by compositions of the following three types of morphisms on the ring level: (1) Localization homomorphisms $R_{\mathfrak{b}} \to R_{\mathfrak{u}}$ and $R^\partial_{\mathfrak{u}} \to R_{\mathfrak{u}}$ (2.18); (2) Automorphisms $\theta_{\mathfrak{p}}$ and $\theta^\partial_{\mathfrak{p}}$ associated to crossing a codimension zero wall (2.19),(2.20); (3) Isomorphisms $\theta_{\mathfrak{j}} : R_{\mathfrak{b}} \to R_{\mathfrak{b}'}$ associated to crossing a codimension one joint (2.22).

Consistency implies that for chambers $\mathfrak{u}$ and $\mathfrak{u}'$ in the same $\sigma \in \mathscr{P}_{\max}$ any two morphisms $U_{\mathfrak{u}} \to U_{\mathfrak{u}'}$ coincide. This follows from a simple topological argument presented in detail in Step 3 of the proof of [GrSi4], Lemma 2.30. The same argument holds for showing uniqueness of the morphism $U^\partial_{\mathfrak{u}} \to U^\partial_{\mathfrak{u}'}$ for boundary chambers $\mathfrak{u}, \mathfrak{u}'$ intersecting the same $\rho \in \mathscr{P}^{[n-1]}$ full-dimensionally. Similarly, for slabs $\mathfrak{b}, \mathfrak{b}'$ contained in the same $\rho \in \mathscr{P}^{[n-1]}$ any two morphisms $U_{\mathfrak{b}} \to U_{\mathfrak{b}'}$ coincide. Finally, for a slab $\mathfrak{b} \subseteq \rho \in \mathscr{P}^{[n-1]}$ and a chamber $\mathfrak{u} \subseteq \sigma \in \mathscr{P}_{\max}$ with $\rho \subseteq \sigma$ all morphisms $U_{\mathfrak{u}} \to U_{\mathfrak{b}}$ agree.

We have thus shown that the full subcategory with exactly one object $U_\rho := U_{\mathfrak{b}}$ for each $\rho \in \mathscr{P}^{[n-1]}$ with $\mathfrak{b} \subseteq \rho$ any slab, one object $U_\sigma := U_{\mathfrak{u}}$ for each $\sigma \in \mathscr{P}_{\max}$ with $\mathfrak{u} \subseteq \sigma$ and one object $U_\rho := \operatorname{Spec} R^\partial_{\mathfrak{u}}$ for each $\rho \in \mathscr{P}^{[n-1]}$, $\rho \subseteq \partial B$, with $\dim \mathfrak{u} \cap \rho = n-1$, defines a skeleton for $\underline{C}$. In particular, whenever $\rho \subseteq \sigma$, we obtain an open embedding $U_\sigma \to U_\rho$. This gives gluing data for the set of schemes $\{U_\rho \,|\, \rho \in \mathscr{P}^{[n-1]}\}$ in the sense of [Hr2], Ex. II.2.12, gluing U_ρ and $U_{\rho'}$ along the open subsets $U_\sigma \subseteq U_\rho, U_{\rho'}$ whenever $\rho, \rho' \subseteq \sigma$, using the identity map on U_σ. The conditions of [Hr2], Ex. II.2.12 are trivially satisfied. Hence one obtains a colimit $\mathfrak{X}^\circ$ of the category $\underline{C}$ in the category of schemes covered by the open sets U_ρ. The remaining properties are then obvious by construction. □

Example 2.17. Consider as B the cone in $\mathbb{R}^2$ generated by $(-1,0)$ and $(1,1)$ with the standard affine structure:

$$B = \mathbb{R}_{\geq 0} \cdot (-1,0) + \mathbb{R}_{\geq 0} \cdot (1,1).$$

Take the polyhedral decomposition with two maximal cells

$$\sigma_1 = \mathbb{R}_{\geq 0} \cdot (-1,0) + \mathbb{R}_{\geq 0} \cdot (0,1), \quad \sigma_2 = \mathbb{R}_{\geq 0} \cdot (0,1) + \mathbb{R}_{\geq 0} \cdot (1,1).$$

We then have one vertex $v = (0,0)$, and three codimension one cells $\rho_1 = \sigma_1 \cap \partial B$, $\rho_2 = \sigma_1 \cap \sigma_2$, $\rho_3 = \sigma_2 \cap \partial B$. Taking $\Delta = \{v\}$ each $\rho_i \setminus \Delta$ is connected. The universal choice of MPA function φ takes values in $\mathbb{N}$ with kink one along the interior edge ρ_2. We can fix a representative of φ which takes the value 0 on σ_1. This gives a splitting of the sheaf $\mathcal{P}$ as the constant sheaf $\mathbb{Z}^2 \oplus \mathbb{Z}$, with the first $\mathbb{Z}^2$ factor being the integral tangent vectors to B. We thus write exponents as elements of $\mathbb{Z}^3$. As final ingredient we take one slab $\mathfrak{b}$ with underlying set ρ_2 and $f_{\mathfrak{b}} = 1 + z^{(0,-1,0)}$. We thus have two chambers $\mathfrak{u}_1 = \sigma_1$ and $\mathfrak{u}_2 = \sigma_2$. We work over $A = \Bbbk$ some field, write $A[Q] = \Bbbk[t]$ and take $I = (t^{k+1})$.

Now $\mathfrak{X}^\circ$ is covered by the spectra of the following three rings, written with $t = z^{(0,0,1)}$, $x = z^{(-1,0,0)}$, $y = z^{(1,1,1)}$, $w = z^{(0,1,0)}$ for readability:

$$\begin{aligned} R^\partial_{\mathfrak{u}_1} &= \Bbbk[x^{\pm 1}, w, t]/(t^{k+1}) \\ R_{\mathfrak{b}} &= \Bbbk[x, y, w^{\pm 1}, t]/(t^{k+1}, xy - (1 + w^{-1})wt) \\ R^\partial_{\mathfrak{u}_2} &= \Bbbk[y^{\pm 1}, w, t], /(t^{k+1}). \end{aligned}$$

To glue we also need the localizations $R_{\mathfrak{u}_1} = (R^\partial_{\mathfrak{u}_1})_w$ and $R_{\mathfrak{u}_2} = (R^\partial_{\mathfrak{u}_2})_w$. In any case, it is not hard to see that $\mathfrak{X}^\circ$ is isomorphic to the complement of the single point $V(X, Y, W, t)$ in the affine scheme

$$\mathfrak{X} = \operatorname{Spec}\big(\Bbbk[X, Y, W, t]/(t^{k+1}, XY - (1 + W)t)\big).$$

If we represent $f \in \Gamma(\mathfrak{X}^\circ, \mathcal{O}_{\mathfrak{X}^\circ})$ by the tuple (f_1, f_2, f_3) of restrictions f_i to $R^\partial_{\mathfrak{u}_1}$, $R_{\mathfrak{b}}$, $R^\partial_{\mathfrak{u}_2}$, the three generators X, Y, W are given by

$$X|_{\mathfrak{X}^\circ} = (x, x, (1+w)y^{-1}t), \quad Y|_{\mathfrak{X}^\circ} = ((1+w)x^{-1}t, y, y), \quad W|_{\mathfrak{X}^\circ} = (w, w, w).$$

These triples of functions are clearly compatible with the gluing morphisms, and they exhibit the relation $XY = (1+W)t$ on each of the three covering affine open sets, hence on $\mathfrak{X}^\circ$.

We will see in §3.4 how X, Y and W are instances of global canonical functions that always exist and that generate the ring R of global functions. Moreover, in the present case of a conical B these functions provide an embedding of $\mathfrak{X}^\circ$ as the complement of a codimension two subset in $\operatorname{Spec} R$. See Example 3.22 for details.

Remark 2.18. If $\partial B \neq \emptyset$ our degeneration $\mathfrak{X}^\circ \to \operatorname{Spec}(A[Q]/I)$ comes with a divisor $\mathfrak{D}^\circ \subseteq \mathfrak{X}^\circ$ as follows. For each boundary chamber $\mathfrak{u}$ we have a monomial ideal in $R^\partial_\mathfrak{u} = (A[Q]/I)[\Lambda_{\sigma,\rho}]$ generated by $\Lambda_{\sigma,\rho} \setminus \Lambda^\times_{\sigma,\rho}$. As observed in the discussion leading to (2.20) these monomial ideals are compatible with the gluing of rings $R_\mathfrak{u}$ for chambers $\mathfrak{u} \subseteq \sigma$. Since these are the only gluings involving boundary chambers intersecting ρ we thus obtain a reduced divisor $\mathfrak{D}^\circ_\rho \subseteq \mathfrak{X}^\circ$.

Moreover, $\mathfrak{D}^\circ$ can be described in the same way as $\mathfrak{X}^\circ$ by a wall structure $\mathscr{S}_\rho$ on ρ, albeit with the codimension one locus removed. To this end define a wall structure $\mathscr{S}_\rho$ by considering those walls $\mathfrak{p} \in \mathscr{S}$ with $\mathfrak{p} \cap \operatorname{Int} \rho \neq \emptyset$. Then each such wall $\mathfrak{p}$ defines the wall with underlying polyhedral set $\mathfrak{p}_\rho := \mathfrak{p} \cap \rho$ and function $f_{\mathfrak{p}_\rho}$ the image of $f_\mathfrak{p}$ under $R_\sigma \to R_\rho$. Here R_ρ is the ring analogous to R_σ associated to the cell ρ of the decomposition $\mathscr{P}_\partial$ of ∂B. Then $\mathscr{S}_\rho$ has no slabs and it is clear that the construction of $\mathfrak{D}^\circ_\rho$ as a closed subscheme of $\mathfrak{X}^\circ$ agrees with the construction by applying the gluing construction to ρ and $\mathscr{S}_\rho$.

Note that the $\mathfrak{D}^\circ_\rho \subseteq \mathfrak{X}^\circ$ are pairwise disjoint and hence $\mathfrak{D}^\circ := \bigcup_{\rho \subseteq \partial B} \mathfrak{D}^\circ_\rho$ defines a closed subscheme of $\mathfrak{X}^\circ$ of codimension one with reduced fibres over $\operatorname{Spec}(A[Q]/I)$.

Reduction of $\mathfrak{X}^\circ$ modulo I_0 yields an open dense subscheme of the scheme X_0 considered in §2.1.

Proposition 2.19. *The reduction of $\mathfrak{X}^\circ$ modulo I_0 is canonically isomorphic to the complement of the union of codimension two strata in X_0. In particular, $\mathfrak{X}^\circ$ is separated as a scheme over $A[Q]/I$.*

Proof. This follows immediately from the construction. □

3. Broken lines and canonical global functions

The main objective in this paper is the construction of a canonical set of globally defined functions on $\mathfrak{X}^\circ$. There is one such function ϑ_m for each asymptotic monomial m on an unbounded cell (Definition 3.1). If X_0 is affine the reduction of the ϑ_m modulo I_0 form a basis of the coordinate ring of X_0 as an $A[Q]/I_0$-module. Hence they can be used to construct a flat affine scheme $\mathfrak{X}$ over $A[Q]/I$ containing $\mathfrak{X}^\circ$ as an open subscheme. In the projective case we apply the procedure to the total space $\mathfrak{L}$ of the inverse of the polarizing line bundle $\mathcal{L}$ to construct a canonical basis of sections of $\mathcal{L}^d$ for any $d \geq 0$. These are the theta functions in the title of the paper.

Throughout the section $\mathscr{S}$ is a wall structure on a polyhedral pseudomanifold $(B, \mathscr{P})$ that is consistent in codimensions zero and one and φ is a convex MPA-function with values in a toric monoid Q with $z^{\kappa_{\underline{\rho}}(\varphi)} \in I_0$ for any $\underline{\rho} \in \tilde{\mathscr{P}}^{[n-1]}$.

Here is the definition of the set of monomials labelling the functions ϑ_m.

Definition 3.1. For a polyhedron $\tau \in \mathscr{P}$ an *asymptotic monomial on* τ is an element $m \in \Gamma(\tau \setminus \Delta, \mathcal{P})$, with $\mathrm{ht}_\sigma(m) = 0$ (Definition 2.8) for each $\sigma \in \mathscr{P}_{\max}$ containing τ, and such that for any $x \in \mathrm{Int}\,\tau$

$$\tau + \mathbb{R}_{\geq 0}\overline{m}_x \subseteq \tau.$$

An asymptotic monomial on a polyhedral pseudomanifold $(B, \mathscr{P})$ is an asymptotic monomial on any $\tau \in \mathscr{P}$. Here we identify asymptotic monomials on different cells via inclusion of faces and extension by parallel transport.

If m is an asymptotic monomial, $\overline{m}$ is called its *tangent vector*.

Note that any $\overline{m} \in \Lambda_x$ has at most one lift to a monomial m with $\mathrm{ht}_\sigma(m) = 0$ for a given $\sigma \in \mathscr{P}_{\max}$ containing x. Hence an asymptotic monomial is uniquely determined by its tangent vector.

For a bounded polyhedron only the zero vector is an asymptotic monomial. In general, writing $\tau = \tau_0 + \tau_\infty$ with τ_0 a bounded polyhedron and τ_∞ the asymptotic cone, the asymptotic monomials on τ are precisely the integral points of τ_∞.

3.1. **Broken lines.** The construction of the canonical function ϑ_m is based on the propagation of monomials along piecewise straight paths. A path can bend when crossing a wall, and the possible new directions of propagation depends on the result of applying the wall crossing isomorphism.

Let $\mathfrak{u}, \mathfrak{u}'$ be adjacent chambers of $\mathscr{S}$. If $\mathfrak{u}$ and $\mathfrak{u}'$ are separated by a codimension zero wall $\mathfrak{p}$ let $\theta_\mathfrak{p}$ be the automorphism of R_σ associated to passing from $\mathfrak{u}$ to $\mathfrak{u}'$ by crossing the wall $\mathfrak{p}$. As a matter of notation we now write $\theta_{\mathfrak{u}'\mathfrak{u}}$ instead of $\theta_\mathfrak{p}$:

$$\theta_{\mathfrak{u}'\mathfrak{u}} := \theta_\mathfrak{p} : R_\mathfrak{u} = R_\sigma \longrightarrow R_\sigma = R_{\mathfrak{u}'}. \tag{3.1}$$

If $\mathfrak{u}$ and $\mathfrak{u}'$ are separated by a slab $\mathfrak{b} \subseteq \underline{\rho}$ let σ, σ' be the maximal cells containing $\mathfrak{u}, \mathfrak{u}'$, respectively. Denote by $R^\mathfrak{b}_\mathfrak{u} \subseteq R_\mathfrak{u} = R_\sigma$ the $A[Q]/I$-subalgebra generated by Λ_ρ and by the image of Z_+ under the localization homomorphism $\chi_{\mathfrak{b},\sigma}$, see (2.18) and (2.21). The conventions are such that Z_+ has tangent vector $\xi(\rho)$ which points from ρ into $\sigma = \sigma(\rho)$. Now define

$$\theta_{\mathfrak{u}'\mathfrak{u}} : R^\mathfrak{b}_\mathfrak{u} \longrightarrow R_{\mathfrak{u}'} \tag{3.2}$$

as follows. Note that $R^\mathfrak{b}_\mathfrak{u}$ is generated as an $A[Q]/I$-algebra by Λ_ρ and by $\chi_{\mathfrak{b},\sigma}(Z_+)$, while $R_{\mathfrak{u}'}$ is generated by Λ_ρ and by $\chi_{\mathfrak{b},\sigma'}(Z_-)^{\pm 1}$. We then define $\theta_{\mathfrak{u}'\mathfrak{u}}$ to be the identity on Λ_ρ and

$$\theta_{\mathfrak{u}'\mathfrak{u}}\big(\chi_{\mathfrak{b},\sigma}(Z_+)\big) = \chi_{\mathfrak{b},\sigma'}(Z_-)^{-1} \cdot f_\mathfrak{b} \cdot z^{\kappa_{\underline{\rho}}}. \tag{3.3}$$

From this one sees easily that if $h \in R^\mathfrak{b}_\mathfrak{u}$, then there exists a unique $\tilde{h} \in (A[Q]/I)[\Lambda_\rho][Z_+] \subseteq R_\mathfrak{b}$ with $\chi_{\mathfrak{b},\sigma}(\tilde{h}) = h$ and $\chi_{\mathfrak{b},\sigma'}(\tilde{h}) = \theta_{\mathfrak{u}'\mathfrak{u}}(h)$.

With $\theta_{\mathfrak{u}'\mathfrak{u}}$ defined for adjacent chambers $\mathfrak{u}, \mathfrak{u}'$ we are now able to propagate certain monomials from $\mathfrak{u}$ to $\mathfrak{u}'$.

Definition 3.2. Let $\mathfrak{u}, \mathfrak{u}'$ be adjacent chambers of $\mathscr{S}$ and $\sigma = \sigma_{\mathfrak{u}}$, $\sigma' = \sigma_{\mathfrak{u}'}$ the maximal cells containing $\mathfrak{u}$ and $\mathfrak{u}'$, respectively. Let az^m, $a \in A[Q]/I$, $m \in \Lambda_x$ for some $x \in \operatorname{Int}\sigma$, be an expression defined at a point of $\operatorname{Int}(\mathfrak{u} \cap \mathfrak{u}')$, using the canonical identification (1.10) on σ, and assume that m points from $\mathfrak{u}'$ to $\mathfrak{u}$.[7] Then in the expansion in $R_{\mathfrak{u}'} = R_{\sigma'}$,

$$\theta_{\mathfrak{u}'\mathfrak{u}}(az^m) = \sum_i a_i z^{m_i} \tag{3.4}$$

with $m_i \in \Lambda_{\sigma_{\mathfrak{u}'}}$ mutually distinct and $a_i \in A[Q]/I$, we call any summand $a_i z^{m_i}$ a *result of transport of* az^m from $\mathfrak{u}$ to $\mathfrak{u}'$.

Note that in the case that $\mathfrak{u}, \mathfrak{u}'$ are separated by a slab $\mathfrak{b}$ the assumption on m implies that $az^m \in R^{\mathfrak{b}}_{\mathfrak{u}}$. It is also important to note that by the definition of $\theta_{\mathfrak{u}'\mathfrak{u}}$ any of the exponents m_i with $a_i \neq 0$ also have the property that m_i points from $\mathfrak{u}'$ to $\mathfrak{u}$.

Next we define the piecewise straight paths carrying propagations of monomials.

Definition 3.3. (Cf. [Gr2], Definition 4.9.) A *broken line* for a wall structure $\mathscr{S}$ on $(B, \mathscr{P})$ is a proper continuous map

$$\beta : (-\infty, 0] \to B_0$$

with image disjoint from any joints of $\mathscr{S}$, along with a sequence $-\infty = t_0 < t_1 < \cdots < t_r = 0$ for some $r \geq 1$ with $\beta(t_i) \in |\mathscr{S}|$ for $i \leq r-1$, and for each $i = 1, \ldots, r$ an expression $a_i z^{m_i}$ with $a_i \in A[Q]/I$, $m_i \in \Lambda_{\beta(t)}$ for any $t \in (t_{i-1}, t_i)$, defined at all points of $\beta([t_{i-1}, t_i])$ (for $i = 1$: $\beta((-\infty, t_1])$), and subject to the following conditions.

(1) $\beta|_{(t_{i-1}, t_i)}$ is a non-constant affine map with image disjoint from $|\mathscr{S}|$, hence contained in the interior of a unique chamber $\mathfrak{u}_i$ of $\mathscr{S}$, and $\beta'(t) = -m_i$ for all $t \in (t_{i-1}, t_i)$.
(2) For each $i = 1, \ldots, r-1$ the expression $a_{i+1} z^{m_{i+1}}$ is a result of transport of $a_i z^{m_i}$ from $\mathfrak{u}_i$ to $\mathfrak{u}_{i+1}$ (Definition 3.2).[8]

Denote by $\sigma \in \mathscr{P}_{\max}$ the cell containing $\beta((-\infty, t_1])$. The broken line is called *normalized* if $a_1 = 1$.

[7] This means precisely that $m \in \Lambda_\sigma$, $\sigma = \sigma_{\mathfrak{u}}$, lies in the half-space generated by tangent vectors pointing from $\mathfrak{u} \cap \mathfrak{u}'$ to $\mathfrak{u}$.

[8] Note that $\beta(t_i) \in \operatorname{Int}(\mathfrak{u} \cap \mathfrak{u}')$ since $\operatorname{im}(\beta)$ is disjoint from joints, so the transport of monomials makes sense.

A broken line with $\beta(0)$ contained in a wall is said to *end on a wall*. The *type* of β is the tuple of all $\mathfrak{u}_i$ and m_i. By abuse of notation we suppress the data t_i, a_i, m_i when talking about broken lines, but introduce the notation

$$a_\beta := a_r, \quad m_\beta := m_r.$$

Remark 3.4. 1) Since $\beta\big((-\infty, t_1]\big)$ is an affine half-line in $B \setminus \Delta$ in direction m_1 it follows that m_1 is an asymptotic monomial (Definition 3.1). We therefore call m_1 the *asymptotic monomial of* β.

2) A normalized broken line β is determined uniquely by its endpoint $\beta(0)$ and its type. In fact, the coefficients a_i are determined inductively from $a_1 = 1$ by Equation (3.4).

3) If $\partial B \neq \emptyset$ it may happen that a broken line has its endpoint $\beta(0)$ on ∂B. By condition (1) in Definition 3.3 the broken line is then maximal, that is, it is not the restriction of another broken line to a proper subset of its domain. Moreover, if $\mathfrak{u} = \mathfrak{u}_r$ is the last chamber visited by β, the monomial z^{m_β} is an element of $R^\partial_\mathfrak{u}$. This follows by the same condition (1) and the definition of $R^\partial_\mathfrak{u} = R_{\sigma,\rho}$, see (2.6).

According to Remark 3.4,2 the map $\beta \mapsto \beta(0)$ identifies the space of broken lines of a fixed type with a subset of $\mathfrak{u}_r$, the last chamber visited by β. This subset is the interior of a polyhedron:

Proposition 3.5. *For each type* $(\mathfrak{u}_i, m_i)$, $i = 1, \ldots, r$, *of broken lines there is a rational closed convex polyhedron* Ξ, *of dimension* n *if non-empty, and an affine immersion*

$$\Phi : \Xi \longrightarrow \mathfrak{u}_r,$$

so that $\Phi\big(\operatorname{Int}\Xi\big)$ *is the set of endpoints* $\beta(0)$ *of broken lines* β *of the given type not ending on a wall.*

Proof. This is an exercise in polyhedral geometry left to the reader. For the statement on dimensions it is important that broken lines are disjoint from joints. □

Remark 3.6. A point $p \in \Phi(\partial\Xi)$ still has a meaning as an endpoint of a piecewise affine map $\beta : (-\infty, 0] \to B$ together with data t_i and $a_i z^{m_i}$, defining a *degenerate broken line*. The point p may correspond to a broken line which ends on a wall, that is, $\beta(0) \in \partial\mathfrak{u}_r$ while $\beta^{-1}(\mathfrak{u}_r)$ contains an open set. Otherwise, this data does not define a broken line, and $\operatorname{im}(\beta)$ has to intersect a joint. Note that by convexity of the chambers, the non-empty intersection with joints comprises the cases that β maps a whole interval to $|\mathscr{S}|$ or that $t_{i-1} = t_i$. All other conditions in the definition of broken lines are open.

By definition, the set of endpoints $\beta(0)$ of degenerate broken lines of a given type is the $(n-1)$-dimensional polyhedral subset $\Phi(\partial\Xi) \subseteq \mathfrak{u}$. The set of endpoints of degenerate broken lines *not transverse* to some joint of $\mathscr{S}$, that is, with an interval mapping to a joint or intersecting the boundary of a joint, is polyhedral of codimension at least two. Thus there is a dense open subset of $\Phi(\partial\Xi)$ of endpoints of degenerate broken lines that are transverse to all joints, but intersecting at least one joint or with endpoint on a wall.

For any fixed asymptotic monomial we have the following finiteness result for types of broken lines.

Lemma 3.7. *For each asymptotic monomial m the set of types of broken lines with asymptotic monomial m is finite.*

Proof. There is a k such that $I_0^k \subseteq I$, since A is assumed to be Noetherian. Let β be a broken line. From (3.3) it follows that if $\beta(t_i)$ lies in a codimension one cell ρ and $a_i z^{m_i} \in I_0^{k'} R_{\sigma_{\mathfrak{u}_i}}$, then $a_{i+1} z^{m_{i+1}} \in z^{\kappa_{\underline{\rho}}} I_0^{k'} R_{\sigma_{\mathfrak{u}_{i+1}}} \subseteq I_0^{k'+1} R_{\sigma_{\mathfrak{u}_{i+1}}}$. Similarly, if $\beta(t_i) \in \operatorname{Int}\sigma$ for some maximal σ, then $\beta(t_i) \in \mathfrak{p}$ for some wall $\mathfrak{p}$, and $f_{\mathfrak{p}} \equiv 1 \mod I_0$. Thus if $m_i \neq m_{i+1}$ and $a_i z^{m_i} \in I_0^{k'} R_\sigma$, we must have $a_{i+1} z^{m_{i+1}} \in I_0^{k'+1} R_\sigma$. Thus any broken line crosses less than k codimension one walls and bends less than k times. Furthermore, the expansion (3.4) is finite, and hence there are at most a finite number of choices for m_{i+1} given m_i. Since every maximal cell in $\mathscr{P}$ contains only a finite number of chambers and walls, it is then clear that the number of types of broken lines for a given asymptotic monomial is finite. □

By Lemma 3.7 and Proposition 3.5 the following definition is meaningful.

Definition 3.8. A point $p \in B$ is called *general* (for the given structure $\mathscr{S}$) if it is not contained in $\Phi(\partial\Xi)$, for any Φ as in Proposition 3.5 for any type of broken line.

3.2. **Consistency and rings in codimension two.** The canonical global functions will be defined on $U_{\mathfrak{u}} = \operatorname{Spec} R_{\mathfrak{u}}$ as a sum of expressions $a_\beta z^{m_\beta}$ over broken lines ending at a point $x \in \operatorname{Int}\mathfrak{u}$. For this definition to lead to a globally well-defined function we need an additional consistency condition, localized at joints of codimension two. We continue to assume that $\mathscr{S}$ is a wall structure on the polyhedral pseudomanifold $(B, \mathscr{P})$ and φ is a convex MPA-function with values in the toric monoid Q with $z^{\kappa_{\underline{\rho}}(\varphi)} \in I_0$ for any $\underline{\rho} \in \tilde{\mathscr{P}}^{[n-1]}$. For the moment we do not impose any consistency assumption in codimensions zero and one.

Let $\mathfrak{j}$ be a joint of codimension two and let $\omega \in \mathscr{P}^{[n-2]}$ be the smallest cell containing $\mathfrak{j}$. Build a new polyhedral pseudomanifold $(B_{\mathfrak{j}}, \mathscr{P}_{\mathfrak{j}})$ by replacing any $\tau \in \mathscr{P}$ with $\tau \supseteq \mathfrak{j}$ by the tangent wedge of ω in τ. Note that the inclusion $\tau \subseteq \tau'$ of faces induces an inclusion of the respective tangent wedges. So $B_{\mathfrak{j}}$ is a local model for $(B, \mathscr{P})$ near $\mathfrak{j}$ all of whose cells are cones. By the S_2 condition on B, in fact $B_{\mathfrak{j}}$ is a manifold (with boundary if $\mathfrak{j}$ is a boundary joint). Moreover, each such cell contains the codimension two linear space $\Lambda_{\mathfrak{j},\mathbb{R}}$. Thus $(B_{\mathfrak{j}}, \mathscr{P}_{\mathfrak{j}})$ is topologically the preimage of a fan in $\mathbb{R}^2$ by a piecewise integral affine submersion $\mathbb{R}^n \to \mathbb{R}^2$. Similarly, the wall structure $\mathscr{S}$ induces a wall structure $\mathscr{S}_{\mathfrak{j}}$ by considering only the walls containing $\mathfrak{j}$ and going over to tangent wedges based at ω for the underlying polyhedral subsets of codimension one. Since the only joint of $\mathscr{S}_{\mathfrak{j}}$ is the codimension two cell $\Lambda_{\mathfrak{j},\mathbb{R}}$ this wall structure is trivially consistent in codimensions zero and one. Denote by $\mathfrak{X}^{\circ}_{\mathfrak{j}}$ the scheme over $A[Q]/I$ constructed in §2.4 for $(B_{\mathfrak{j}}, \mathscr{P}_{\mathfrak{j}})$.

Now let m be an asymptotic monomial on $(B_{\mathfrak{j}}, \mathscr{P}_{\mathfrak{j}})$. For a general point $p \in B_{\mathfrak{j}}$, say contained in the chamber $\mathfrak{u}$ for $\mathscr{S}_{\mathfrak{j}}$, define

$$\vartheta^{\mathfrak{j}}_m(p) := \sum_{\beta} a_\beta z^{m_\beta} \in R_{\mathfrak{u}}. \tag{3.5}$$

The sum runs over all normalized broken lines on $(B_{\mathfrak{j}}, \mathscr{P}_{\mathfrak{j}})$ with asymptotic monomial m and endpoint p.

Definition 3.9. The wall structure $\mathscr{S}$ is *consistent along the codimension two joint* $\mathfrak{j}$ if the $\vartheta^{\mathfrak{j}}_m(p)$ (a) do not depend on the choice of general point p in the same chamber $\mathfrak{u}$ and (b) are compatible with the change of chambers morphisms $\theta_{\mathfrak{u}'\mathfrak{u}}$ for $\mathscr{S}_{\mathfrak{j}}$ defined in (3.1) and (3.2).

A wall structure $\mathscr{S}$ is *consistent* if it is consistent in codimensions zero, one (Definitions 2.13 and 2.14) and along each codimension two joint.

Remark 3.10. Consistency at a joint $\mathfrak{j}$ can be reduced to the two-dimensional case as follows. Denote by $\overline{\overline{B}}_{\mathfrak{j}}$ the image of $B_{\mathfrak{j}}$ under the piecewise integral affine submersion $B_{\mathfrak{j}} \to \mathbb{R}^2$ that contracts $\mathfrak{j}$ to the origin. If $\mathfrak{p} \in \mathscr{S}_{\mathfrak{j}}$ is a wall denote its image in $\overline{\overline{B}}_{\mathfrak{j}}$ by $\overline{\overline{\mathfrak{p}}}$. By extending the base ring from $A[Q]$ to $A[Q \oplus \Lambda_{\mathfrak{j}}]$ the function $f_{\mathfrak{p}}$ attached to $\mathfrak{p}$ can be interpreted as a function attached to $\overline{\overline{\mathfrak{p}}}$, thus endowing $\overline{\overline{B}}_{\mathfrak{j}}$ with a wall structure $\overline{\overline{\mathscr{S}}}_{\mathfrak{j}}$. Then there is a one-to-one correspondence between broken lines β for $\mathscr{S}_{\mathfrak{j}}$ with asymptotic direction not contained in $\Lambda_{\mathfrak{j}}$ with fixed endpoint p and broken lines for $\overline{\overline{\mathscr{S}}}_{\mathfrak{j}}$ with fixed endpoint the image of p in $\overline{\overline{B}}_{\mathfrak{j}}$.

Example 3.11. Continuing with Example 2.15,2, we noted that an arbitrary scattering diagram on the pair (B, Σ) arising from a pair (Y, D) provides a wall structure consistent in codimensions zero and one. The only joint in codimension two is $\mathfrak{j} = \{0\}$, and $(B_{\mathfrak{j}}, \Sigma_{\mathfrak{j}}) = (B, \Sigma)$. It is highly non-trivial to construct a wall structure which is consistent in codimension two; in fact, the construction of such a wall structure can be viewed as the main result of [GHK1]. In particular, Definition 3.3 of [GHK1] defines the *canonical scattering diagram* which gives a wall structure of the current paper as in Example 2.15,2. This data, motivated by [GPS], is determined by certain relative Gromov-Witten invariants of (Y, D). The definition of this diagram requires the choice of the monoid Q and multi-valued function φ. As in Example 2.1, one chooses a monoid Q containing $\mathrm{NE}(Y)$. The function φ is chosen to have kink $\kappa_{\rho_i}(\varphi)$ the class $[D_i] \in H_2(Y, \mathbb{Z})$ of the irreducible component of D corresponding to ρ_i. If Q is chosen so that $Q^\times = 0$, it follows from [GHK1], Theorem 3.8 that the canonical scattering diagram is consistent in codimension two.

Again if we choose a set $\bar{B} \subseteq B$ a compact two-dimensional subset containing the origin, the canonical scattering diagram gives a wall structure on $(\bar{B}, \mathscr{P})$ as in Example 2.15,2. It has only one interior codimension two joint $\mathfrak{j} = \{0\}$, and $(\bar{B}_\mathfrak{j}, \mathscr{P}_\mathfrak{j}) = (B, \Sigma)$. From the previous paragraph, it follows that the wall structure on $(\bar{B}, \mathscr{P})$ induced by the canonical scattering diagram on (B, Σ) is consistent along this interior codimension two joint.

Consistency at the boundary joints is more subtle. In the present case with all monomials outgoing, that is, extending to the compactifying divisor, consistency is equivalent to local convexity of $\bar{B}$ along the boundary, see Proposition 3.13. One can show [GHK2] that $\bar{B}$ with locally convex boundary exists if and only if the divisor D supports a big and nef divisor for Y.

At an interior joint consistency poses a condition on the behaviour of broken lines when crossing the joint. In contrast, at a boundary joint the question is about sufficient local convexity of the boundary to balance the incoming monomials on the walls containing $\mathfrak{j}$, see Proposition 3.13 below. To formulate this convexity condition recall that by Remark 3.10 we may restrict to $\dim B_\mathfrak{j} = 2$. Because the only singular point is the origin, $B_\mathfrak{j}$ can be embedded into $\mathbb{R}^2$ as a not necessarily convex cone containing $\mathbb{R}_{>0} \cdot (0, 1)$ in its interior and with boundary $\mathbb{R}_{\geq 0} \cdot (-1, 0) \cup \mathbb{R}_{\geq 0} \cdot (a, b)$, $a > 0$. Denote the walls (of codimensions 0 and 1) not contained in ∂B by $\mathfrak{p}_j = \mathbb{R}_{\geq 0} \overline{m}_j$, $j = 1, \ldots, r$, ordered clockwise and with $\overline{m}_j = (a_j, b_j)$ primitive. Any monomial in the function $f_{\mathfrak{p}_j}$ has tangent vector $-\delta \overline{m}_j$ for $\delta \in \mathbb{Z}$. Let δ_j be the maximum of the δ occurring in $f_{\mathfrak{p}_j}$. As we will see in the proof of Proposition 3.13 a broken line approaching $\mathfrak{p}_j$ in direction $(1, \lambda_j)$ and maximally bent away from the boundary leaves $\mathfrak{p}_j$ in direction

$$\big(1 + (b_j - a_j\lambda_j)\delta_j a_j, \lambda_j + (b_j - a_j\lambda_j)\delta_j b_j\big). \tag{3.6}$$

This computation motivates us to define $\lambda_j \in \mathbb{Q}$ for $j \geq 0$ inductively by $\lambda_0 := 0$ and

$$\lambda_{j+1} := \frac{\lambda_j + (b_j - a_j\lambda_j)\delta_j b_j}{1 + (b_j - a_j\lambda_j)\delta_j a_j}. \tag{3.7}$$

Definition 3.12. The wall structure $\mathscr{S}$ is called *convex at a boundary joint* $\mathfrak{j} \subseteq \partial B$ if $\mathbb{R}_{\geq 0} \cdot (1, \lambda_r) \not\subseteq \mathrm{Int}(B_{\mathfrak{j}})$.

This notion of convexity at a boundary joint $\mathfrak{j}$ a priori depends on the choice of orientation of the normal space to $\mathfrak{j}$. The recursive equation (3.7) for λ_j, however, is equivalent to

$$\lambda_j = \frac{\lambda_{j+1} + (-b_j + a_j\lambda_{j+1})\delta_j b_j}{1 + (-b_j + a_j\lambda_{j+1})\delta_j a_j},$$

which agrees with the change of slope when approaching the wall $\mathfrak{p}_j$ from the other side.

Proposition 3.13. *The wall structure $\mathscr{S}$ is consistent at a joint $\mathfrak{j} \subseteq \partial B$ if it is convex at $\mathfrak{j}$.*

Proof. Let us first verify the claim above that (3.7) describes the maximal change of slope away from ∂B of a broken line when passing through the wall $\mathfrak{p}_j$. Let $(c, d) \in \mathbb{Z}^2$ be the tangent vector of the monomial z^m of the broken line before hitting $\mathfrak{p}_j$. Then the result of transport through $\mathfrak{p}_j$ selects a monomial of $f_{\mathfrak{p}_j}^{-b_j c + a_j d} \cdot z^m$. The tangent vector of such a monomial is of the form

$$\mu\overline{m}_j + (c, d) = c \cdot \Big(1 + \frac{\mu}{c} a_j, \frac{d}{c} + \frac{\mu}{c} b_j\Big),$$

with

$$\mu \geq -\delta_j(-b_j c + a_j d) \tag{3.8}$$

an integer. Putting $\lambda_j = d/c$ and $\mu/c = \delta_j(b_j - a_j\lambda_j)$ gives (3.6).

To prove the proposition observe that each type of broken line β on $B_{\mathfrak{j}}$ with asymptotic monomial m has its endpoint in a chamber $\mathfrak{u}_j$, one of the cones $\mathbb{R}_{\geq 0} \cdot \overline{m}_j + \mathbb{R}_{\geq 0} \cdot \overline{m}_{j+1}$, $j = 0, \ldots, r$. To cover the cases $j = 0$ and $j = r$ we define $\overline{m}_0 := (-1, 0)$ and $\overline{m}_{r+1} := (a, b)$.

We now consider two cases for broken lines. The first case is that the monomial z^{m_β} at the endpoint of β has $-\overline{m}_\beta \in \mathrm{Int}\, \mathfrak{u}_j$. In this case, the sum defining $\vartheta^{\mathfrak{j}}_m(p)$ loses one term when p is moved across the ray $-\mathbb{R}_{\geq 0} \cdot \overline{m}_\beta$ in $\mathfrak{u}_j$, with p moving from the side of the ray containing the asymptotic direction m. Indeed, if, say, the asymptotic direction m lies in $\mathbb{R}_{\geq 0}\overline{m}_0 - \mathbb{R}_{\geq 0}\overline{m}_\beta$, the last segment of β must begin on the ray $\mathbb{R}_{\geq 0}\overline{m}_j$ and hence lie in the cone $\mathbb{R}_{\geq 0}\overline{m}_j - \mathbb{R}_{\geq 0}\overline{m}_\beta$. The second case is that the monomial z^{m_β} at the endpoint of β does not satisfy $-\overline{m}_\beta \in B$. Then for $j < r$ any broken line of this type can be extended until it hits $\mathfrak{p}_{j+1}$, and then $\vartheta^{\mathfrak{j}}_m(p)$ is compatible with the change of chambers morphism $\theta_{\mathfrak{p}_{j+1}}$. For $j = r$ there is no further wall to be considered.

The upshot of this discussion is that consistency fails if there is a type of broken line where the closure of the cone of endpoints does not fill the last chamber $\mathfrak{u}_r$.

By monotonicity of λ_{j+1} as a function of λ_j (noting that $\partial\lambda_{j+1}/\partial\lambda_j$ is easily seen to be non-negative) this is the case if it holds for the extreme cases of broken lines with asymptotic monomial $(-1,0)$ or (a,b) and maximal possible bend, that is, where the inequality (3.8) is an equality for each j. By symmetry it suffices to consider the first case. Thus we consider a type of broken lines β with $\beta'(t)$ positively proportional to $(1,\lambda_j)$ inside the chamber $\mathfrak{u}_j$. The convexity condition $\mathbb{R}_{\geq 0}\cdot(1,\lambda_r)\not\subseteq \mathrm{Int}(B_{\mathfrak{j}})$ implies that the endpoints of broken lines of this type fill the chamber containing $\mathbb{R}_{\geq 0}\cdot(a,b)$. Thus consistency at $\mathfrak{j}$ is implied by convexity of $\mathscr{S}$ at $\mathfrak{j}$. □

Remark 3.14. Consistency at a joint $\mathfrak{j}\subseteq\partial B$ only fails to be equivalent to convexity at $\mathfrak{j}$ because there may be no broken line for which the inequality (3.8) is in fact an equality. This is because a product of coefficients in $f_{\mathfrak{p}_j}$ making up a term in $f_{\mathfrak{p}_j}^{-b_jc+a_jd}$ may in fact be 0 as the coefficients lie in $A[Q]/I$, which has nilpotents. However, if one is interested in working over the formal completion $\widehat{A[Q]}$ of $A[Q]$ with respect to the ideal I_0, then this problem disappears. This is not particularly satisfactory as δ_j may no longer exist as $f_{\mathfrak{p}_j}$ is now a formal power series, but in many cases, it is reasonable to check consistency by hand using the proof of Proposition 3.13

Example 3.15. The convexity notion of Definition 3.12 does not imply that B is convex in the usual sense: the wall structure $\mathscr{S}$ can "repair" a non-convex boundary point. For example, let B be the union of the two cones

$$\sigma_1=\mathbb{R}_{\geq 0}\cdot(-1,0)+\mathbb{R}_{\geq 0}\cdot(0,1),\quad \sigma_2=\mathbb{R}_{\geq 0}\cdot(0,1)+\mathbb{R}_{\geq 0}\cdot(1,-1).$$

Let $\rho=\mathbb{R}_{\geq 0}\cdot(0,1)$. We can take φ to take the values 0 at $(-1,0)$ and $(0,1)$ and the value 1 at $(1,-1)$ Finally, we take a structure $\mathscr{S}=\{(\rho_2,z^{(0,1,0)})\}$. It is easy to see this satisfies our modified definition of convexity.

Applying our construction to this data in fact gives the same result as applying it to $B=\sigma_1\cup\sigma_2$ with σ_1 as before and σ_2 the first quadrant, with $\mathscr{S}$ empty.

The point of the definition of consistency in codimension two is that the $\vartheta_m^{\mathfrak{j}}(p)$ now patch to regular functions on $\mathfrak{X}_{\mathfrak{j}}^\circ$, as we will now show. The analogous global statement is the content of Theorem 3.19 below.

Proposition 3.16. *Assume that $\mathscr{S}$ is consistent along the codimension two joint $\mathfrak{j}$ (Definition 3.9). Then for an asymptotic monomial m on $(B_{\mathfrak{j}},\mathscr{P}_{\mathfrak{j}})$ there is a function $\vartheta_m^{\mathfrak{j}}$ on $\mathfrak{X}_{\mathfrak{j}}^\circ$ that restricts to $\vartheta_m^{\mathfrak{j}}(p)\in R_{\mathfrak{u}}$ at any general point p of a chamber $\mathfrak{u}$.*

Proof. Condition (a) in Definition 3.9 implies that for any chamber $\mathfrak{u}$ of $\mathscr{S}_\mathfrak{j}$ there is a well-defined element $\vartheta^\mathfrak{j}_m(\mathfrak{u}) \in R_\mathfrak{u}$. Then (b) means that for chambers $\mathfrak{u}, \mathfrak{u}'$ of $\mathscr{S}_\mathfrak{j}$ separated by a codimension zero wall $\mathfrak{p}$ it holds $\vartheta^\mathfrak{j}_m(\mathfrak{u}') = \theta_\mathfrak{p}\big(\vartheta^\mathfrak{j}_m(\mathfrak{u})\big)$.

If $\mathfrak{u}, \mathfrak{u}'$ are separated by a slab $\mathfrak{b}$ we claim the existence of an element $\vartheta^\mathfrak{j}_m(\mathfrak{b}) \in R_\mathfrak{b}$ with $\vartheta^\mathfrak{j}_m(\mathfrak{u}) = \chi_{\mathfrak{b},\sigma}\big(\vartheta^\mathfrak{j}_m(\mathfrak{b})\big)$, $\vartheta^\mathfrak{j}_m(\mathfrak{u}') = \chi_{\mathfrak{b},\sigma'}\big(\vartheta^\mathfrak{j}_m(\mathfrak{b})\big)$ for $\sigma, \sigma' \in \mathscr{P}_{\max}$ the maximal cells containing $\mathfrak{u}, \mathfrak{u}'$, respectively. By injectivity of the diagonal map $R_\mathfrak{b} \to R_\sigma \times R_{\sigma'}$ the element $\vartheta^\mathfrak{j}_m(\mathfrak{b})$ is unique if it exists.

To show existence consider the set of possibly degenerate broken lines which end on the wall $\mathfrak{b}$, (that is, $\beta(0) \in \mathfrak{b}$) but with $\beta((-\epsilon, 0]) \not\subseteq \mathfrak{b}$ for any $\epsilon > 0$. Similarly to Proposition 3.5 there is a finite union $A \subseteq \mathfrak{b}$ of rational polyhedral subsets of dimension at most $n-2$ such that any such degenerate broken line (see Remark 3.6) with asymptotic monomial m ending at $\mathfrak{b} \setminus A$ is in fact a broken line. Fix $p \in \mathfrak{b} \setminus A$. The set of broken lines β with final segment not contained in $\mathfrak{b}$ and with asymptotic monomial m and $\beta(0) = p$ decomposes as a disjoint union $\mathfrak{B}_I \sqcup \mathfrak{B}_{II}$ depending if β maps the last interval $(t_{r-1}, 0)$ into (I) $\operatorname{Int} \mathfrak{u}$ or (II) $\operatorname{Int} \mathfrak{u}'$. Let $\vartheta^\parallel_m$ be the sum of terms in either $\vartheta^\mathfrak{j}_m(\mathfrak{u})$ or $\vartheta^\mathfrak{j}_m(\mathfrak{u}')$ lying in $(A[Q]/I)[\Lambda_\mathfrak{b}]$. By the definition of consistency and of $\theta_{\mathfrak{u}'\mathfrak{u}}$, $\vartheta^\parallel_m$ is well-defined. We now define $\vartheta^\mathfrak{j}_m(\mathfrak{b})$ as an element of $R_\mathfrak{b}$ by

$$\vartheta^\mathfrak{j}_m(\mathfrak{b}) := \sum_{\beta \in \mathfrak{B}_I} a_\beta z^{m_\beta} + \sum_{\beta \in \mathfrak{B}_{II}} a_\beta z^{m_\beta} + \vartheta^\parallel_m,$$

with $a_\beta \in A[Q]/I$ and the individual monomials interpreted as follows. Let $\xi = \xi(\rho) \in \Lambda_p$ be the chosen generator of Λ_p/Λ_ρ, assumed without loss of generality to point into $\mathfrak{u}$. Then for $\beta \in \mathfrak{B}_I$ the exponent m_β can be written as $a\xi + m'_\beta$ with $a > 0$ and $m'_\beta \in \Lambda_\rho$. Now interpret z^{m_β} as the monomial $Z_+^a \cdot z^{m'_\beta}$ in $R_\mathfrak{b}$, which is the unique lift of $z^{m_\beta} \in R_\mathfrak{u}$ of the stated form under the localization homomorphism $\chi_{\mathfrak{b},\sigma}$. Similarly, for $\beta \in \mathfrak{B}_{II}$ there is a unique lift of $z^{m_\beta} \in R_{\mathfrak{u}'}$ of the form $Z_-^a \cdot z^{m'_\beta}$. Finally, $\vartheta^\parallel_m$ is interpreted as an element of $R_\mathfrak{b}$ via the inclusion $(A[Q]/I)[\Lambda_\rho] \subseteq R_\mathfrak{b}$. This maps to $\theta^\parallel_m \in R_\mathfrak{u}, R_{\mathfrak{u}'}$ under the respective localizations $\chi_{\mathfrak{b},\sigma}$, $\chi_{\mathfrak{b},\sigma'}$.

Moving p into $\mathfrak{u}$, a broken line in $\mathfrak{B}_I$ deforms uniquely without changing $a_\beta z^{m_\beta}$. If $\beta \in \mathfrak{B}_{II}$ it follows from the definition of the transport of monomials through slabs that a broken line splits into several broken lines according to the expansion of $\chi_{\mathfrak{b},\sigma}(Z_-^a \cdot z^{m'_\beta})$ into monomials. This shows $\chi_{\mathfrak{b},\sigma}(\vartheta^\mathfrak{j}_m(\mathfrak{b})) = \vartheta^\mathfrak{j}_m(\mathfrak{u})$. A similar discussion holds with $\mathfrak{u}'$ replacing $\mathfrak{u}$. This proves the claim on existence of $\vartheta^\mathfrak{j}_m(\mathfrak{b})$.

If $\mathfrak{j}$ is a consistent boundary joint and $\mathfrak{u}$ is a boundary chamber we observed in the proof of Proposition 3.13 that each broken line with endpoint in $\mathfrak{u}$ extends to ∂B. This implies that the tangent vector of m_β points from ∂B into B. This shows that z^{m_β} in fact lies in the subring $R^\partial_\mathfrak{u} \subseteq R_\mathfrak{u}$ defined in (2.16).

Summarizing, if $\mathfrak{j}$ is a consistent joint then the $\vartheta^{\mathfrak{j}}_m(\mathfrak{u})$, $\vartheta^{\mathfrak{j}}_m(\mathfrak{b})$ glue to a global regular function on $\mathfrak{X}^{\circ}_{\mathfrak{j}}$. □

Proposition 3.17. *In the situation of Proposition 3.16 the $\vartheta^{\mathfrak{j}}_m$ freely generate the $A[Q]/I$-algebra*

$$R_{\mathfrak{j}} := \Gamma(\mathfrak{X}^{\circ}_{\mathfrak{j}}, \mathcal{O}_{\mathfrak{X}^{\circ}_{\mathfrak{j}}})$$

of global functions on $\mathfrak{X}^{\circ}_{\mathfrak{j}}$ as an $A[Q]/I$-module. In particular, $R_{\mathfrak{j}}$ is flat over $A[Q]/I$. Moreover, the canonical map

$$\mathfrak{X}^{\circ}_{\mathfrak{j}} \longrightarrow \mathfrak{X}_{\mathfrak{j}} := \operatorname{Spec}(R_{\mathfrak{j}}).$$

is an open embedding.

Proof. Proposition 2.19 and the description of X_0 in Proposition 2.2 imply that the statements are true modulo I_0. Denote by $X_{\mathfrak{j},0}$ the flat $A[Q]/I_0$-scheme obtained from $B_{\mathfrak{j}}$ via (2.2). See Proposition 2.2 for an explicit description. Following the proof of [GHK1], Theorem 2.28, consider the ringed space $\mathfrak{X}_{\mathfrak{j}}$ with underlying topological space $|X_{\mathfrak{j},0}|$ and sheaf of $A[Q]/I$-algebras

$$\mathcal{O}_{\mathfrak{X}_{\mathfrak{j}}} := i_*\mathcal{O}_{\mathfrak{X}^{\circ}_{\mathfrak{j}}}$$

where $i : |X^{\circ}_{\mathfrak{j},0}| \to |X_{\mathfrak{j},0}|$ is the inclusion. By the existence of the functions $\vartheta^{\mathfrak{j}}_m$ the reduction morphism $\mathcal{O}_{\mathfrak{X}_{\mathfrak{j}}} \to \mathcal{O}_{X_{\mathfrak{j},0}}$ modulo I_0 is surjective. Thus by [GHK1], Lemma 2.29, $\mathfrak{X}_{\mathfrak{j}}$ is flat over $A[Q]/I$. While [GHK1] only discusses the two-dimensional case, the proof of the cited lemma holds literally in all dimensions provided Lemma 2.10 in [GHK1] in the proof is replaced by Lemma 3.18 below. Moreover, $\mathfrak{X}_{\mathfrak{j}}$ as an infinitesimal extension of the affine scheme $X_{\mathfrak{j},0}$ is itself affine. Now [GHK1], Lemma 2.30, shows that the $\vartheta^{\mathfrak{j}}_m$ are an $A[Q]/I$-module basis of $\Gamma(\mathfrak{X}_{\mathfrak{j}}, \mathcal{O}_{\mathfrak{X}_{\mathfrak{j}}})$.

By the same token, $\operatorname{Spec}(R_{\mathfrak{j}})$ is a flat deformation of $X_{\mathfrak{j},0}$ with the same $A[Q]/I$-module basis for the ring of global regular functions. Thus the embedding $\mathfrak{X}^{\circ}_{\mathfrak{j}} \to \mathfrak{X}_{\mathfrak{j}}$ induces an isomorphism

$$\Gamma(\mathfrak{X}_{\mathfrak{j}}, \mathcal{O}_{\mathfrak{X}_{\mathfrak{j}}}) \longrightarrow R_{\mathfrak{j}} = \Gamma(\mathfrak{X}^{\circ}_{\mathfrak{j}}, \mathcal{O}_{\mathfrak{X}^{\circ}_{\mathfrak{j}}}).$$

In particular, $\mathfrak{X}_{\mathfrak{j}} = \operatorname{Spec}(R_{\mathfrak{j}})$ and $R_{\mathfrak{j}}$ is freely generated by the $\theta^{\mathfrak{j}}_m$. □

In the proof we used the following technical lemma generalizing [Hk], Lemma A.3.

Lemma 3.18. *Let $X \to S$ be a flat family of schemes of pure dimension n such that each fibre X_s satisfies Serre's S_2 condition. Let $U \subseteq X$ be an open subset such that $\dim (X \setminus U) \cap X_s \leq n-2$ for all $s \in S$. Then if $i : U \hookrightarrow X$ is the inclusion, the canonical map $\mathcal{O}_X \to i_*i^*\mathcal{O}_X$ is an isomorphism.*

Proof. By [Gt2], Proposition 5.11.1, the sheaf $i_*i^*\mathcal{O}_X$ is coherent. Let K and C be the kernel and cokernel of the canonical map $\mathcal{O}_X \to i_*i^*\mathcal{O}_X$. These are supported on closed subsets Z_K and Z_C of $X \setminus U$ respectively. Let $Z = Z_K$ or Z_C. We assume Z is non-empty, so that we can choose a generic point p of $Z_s = Z \cap X_s$ for some $s \in S$. Necessarily the closure of $\{p\}$ is a closed subset of X_s of codimension ≥ 2. So by the S_2 condition, there is a regular sequence $x_s, y_s \in \mathfrak{m}_{X_s,p}$ for $\mathcal{O}_{X_s,p}$. By assumption that p is a generic point of Z_s, we can replace x_s, y_s with x_s^ν, y_s^ν for some $\nu \gg 0$ and assume x_s, y_s lie in the ideal of Z_s in the local ring $\mathcal{O}_{X_s,p}$. By [Ma], Theorem 16.1, x_s, y_s is still a regular sequence for $\mathcal{O}_{X_s,p}$. We can lift x_s, y_s to elements of the ideal of Z in $\mathcal{O}_{X,p}$, so that x, y is a regular sequence for $\mathcal{O}_{X,p}$ (see [Ma], pg. 177, Cor. to Theorem 22.5). Equivalently, we have an exact sequence

$$0 \to \mathcal{O}_{X,p} \overset{(y,-x)}{\longrightarrow} \mathcal{O}_{X,p} \oplus \mathcal{O}_{X,p} \overset{(x,y)}{\longrightarrow} \mathcal{O}_{X,p}.$$

Now consider first the case $Z = Z_K$. Since Z_K is the support of K, any given element of K is annihilated by some power of the ideal $\mathcal{I}_Z \subseteq \mathcal{O}_X$. So since $K_p \neq 0$, there exists a non-zero element $g \in K_p$ such that $\mathcal{I}_Z g = 0$ locally at p. Then $xg = yg = 0$, contradicting the exactness of the above sequence. Thus $Z_K = \emptyset$. Similarly, take $Z = Z_C$. Then there is a $g \in (i_*i^*\mathcal{O}_X)_p \setminus \mathcal{O}_{X,p}$ such that $\mathcal{I}_Z g \subseteq \mathcal{O}_{X,p}$. Again using the exact sequence above, since $(yg, -xg) \mapsto 0$ under the second map, we obtain $(yg, -xg) = (yg', -xg')$ for some $g' \in \mathcal{O}_{X,p}$. But then $g = g'$, a contradiction. Thus $Z_C = \emptyset$. □

3.3. **The canonical global functions ϑ_m.** We now give the construction of the canonical global functions ϑ_m in the general case.

Theorem 3.19. *Let $\mathscr{S}$ be a consistent wall structure on the polyhedral pseudomanifold $(B, \mathscr{P})$, and let $\mathfrak{X}^\circ$ be the corresponding flat scheme over $A[Q]/I$ (Proposition 2.16).*

Then for each asymptotic monomial m (Definition 3.1) there exists a function $\vartheta_m \in \Gamma(\mathfrak{X}^\circ, \mathcal{O}_{\mathfrak{X}^\circ})$ restricting on $R_\mathfrak{u}$, $\mathfrak{u}$ a chamber of $\mathscr{S}$, to the sum

$$\vartheta_m(p) := \sum_\beta a_\beta z^{m_\beta}. \tag{3.9}$$

over normalized broken lines with asymptotic monomial m and ending at a general point $p \in \mathfrak{u}$. Moreover, the ϑ_m form an $A[Q]/I$-module basis of $\Gamma(\mathfrak{X}^\circ, \mathcal{O}_{\mathfrak{X}^\circ})$:

$$\Gamma(\mathfrak{X}^\circ, \mathcal{O}_{\mathfrak{X}^\circ}) = \bigoplus_m \big(A[Q]/I\big) \cdot \vartheta_m.$$

Proof. Without joints contained in the singular locus Δ of the affine structure compatibility of $\vartheta_m(p)$ with varying p within a chamber is contained in [CPS], Lemma 4.7. This proof works literally the same in the present case with the assumption of consistency in codimension two. Here Proposition 3.16 replaces [CPS], Proposition 3.2 at codimension two joints. This latter proposition describes the result of transporting a monomial across $B_\mathfrak{j}$ for a joint $\mathfrak{j}$, in the context of [GrSi4] with Δ transverse to joints, in particular defining the local canonical functions $\vartheta_m^\mathfrak{j}$.

To see that the ϑ_m just defined generate $\Gamma(\mathfrak{X}^\circ, \mathcal{O}_{\mathfrak{X}^\circ})$, denote by $\mathcal{I}_0$ the pullback of the ideal $I_0 \subseteq A[Q]/I$ to $\mathfrak{X}^\circ$, and let X_k° be the closed subscheme of $\mathfrak{X}^\circ$ defined by the ideal $\mathcal{I}_0^k$. We will show inductively on k that the ϑ_m form an $A[Q]/(I+I_0^{k+1})$-module basis of $\Gamma(X_k^\circ, \mathcal{O}_{X_k^\circ})$. For sufficiently large k, $I_0^k \subseteq I$ since $\sqrt{I} = I_0$, so we conclude the result for $\mathfrak{X}^\circ$.

For the $k = 0$ case, X_0° is the complement in X_0 of the union of toric strata of codimension two, see Proposition 2.19. In this case the statement follows from standard toric geometry over $A[Q]/I_0$.

Suppose the result is true for $k-1$ with $k \geq 1$. By flatness, there is a short exact sequence

$$0 \longrightarrow (I + I_0^k)/(I + I_0^{k+1}) \otimes \mathcal{O}_{X_0^\circ} \longrightarrow \mathcal{O}_{X_k^\circ} \longrightarrow \mathcal{O}_{X_{k-1}^\circ} \longrightarrow 0$$

of abelian sheaves on X_0°, see [Ma], Theorem 22.3. Taking global sections gives the following exact sequence of $A[Q]$-modules:

(3.10)

$$0 \longrightarrow (I + I_0^k)/(I + I_0^{k+1}) \otimes \Gamma(X_0^\circ, \mathcal{O}_{X_0^\circ}) \longrightarrow \Gamma(X_0^\circ, \mathcal{O}_{X_k^\circ}) \longrightarrow \Gamma(X_0^\circ, \mathcal{O}_{X_{k-1}^\circ}).$$

By the induction hypothesis, the ϑ_m form an $A[Q]/(I+I_0^k)$-basis for $\Gamma(X_0^\circ, \mathcal{O}_{X_{k-1}^\circ})$. Thus given any $s \in \Gamma(X_0^\circ, \mathcal{O}_{X_k^\circ})$, the image of s in $\Gamma(X_0^\circ, \mathcal{O}_{X_{k-1}^\circ})$ can be written as a finite sum $\sum_i \bar{c}_i \vartheta_{m_i}$ with $\bar{c}_i \in A[Q]/(I + I_0^k)$. Lifting each $\bar{c}_i$ to $c_i \in A[Q]/(I + I_0^{k+1})$, we have that $s' = \sum_i c_i \vartheta_{m_i} \in \Gamma(X_0^\circ, \mathcal{O}_{X_k^\circ})$ has the same image as s in $\Gamma(X_0^\circ, \mathcal{O}_{X_{k-1}^\circ})$. Hence $s - s' \in (I + I_0^k)/(I + I_0^{k+1}) \otimes \Gamma(X_0^\circ, \mathcal{O}_{X_0^\circ})$, which by the base case can be written as a sum $\sum_j d_j \vartheta_{m_j'}$ with $d_j \in I + I_0^k$. Thus s itself can be written as a linear combination of theta functions.

Linear dependence is shown similarly: if $\sum c_i \vartheta_{m_i} = 0$ in $\Gamma(X_0^\circ, \mathcal{O}_{X_k^\circ})$, then by the induction hypothesis $c_i \in I + I_0^k$ for each i, and by the base case $c_i = 0$. □

3.4. **The conical case.** A particular case arises when all cells of $\mathscr{P}$ are cones ("conical"). Then $\mathscr{P}$ has exactly one vertex, and this vertex is the only bounded cell. On the scheme-theoretic side the condition means that X_0 is affine. In the most general situation we will want to construct theta functions as sections of a line bundle using the cone over B, so conical pseudomanifolds play a crucial role in the most general construction. It therefore seems appropriate to develop the conical case here before treating the most general case.

Definition 3.20. A polyhedral pseudomanifold $(B,\mathscr{P})$ is called *conical* if each element of $\mathscr{P}$ is a cone. A conical polyhedral pseudomanifold has a single vertex v. A wall structure $\mathscr{S}$ on a conical polyhedral pseudomanifold is called *conical* if each wall $\mathfrak{p}$ in $\mathscr{S}$ is a cone with vertex v.

Assume now that $\mathscr{S}$ is a conical wall structure on the conical polyhedral pseudomanifold $(B,\mathscr{P})$ that is consistent. Then by Theorem 3.19 for each asymptotic monomial ϑ_m we have one distinguished global function ϑ_m on $\mathfrak{X}^\circ$. In the present conical case these functions provide an embedding of $\mathfrak{X}^\circ$ into an affine scheme with the complement of the image of codimension at least two.

Proposition 3.21. *Let $\mathscr{S}$ be a consistent wall structure on the conical polyhedral pseudomanifold $(B,\mathscr{P})$ and let $\mathfrak{X}^\circ$ be the associated scheme over $A[Q]/I$. Then the ϑ_m freely generate $R := \Gamma(\mathfrak{X}^\circ, \mathcal{O}_{\mathfrak{X}^\circ})$ as an $A[Q]/I$-module, and the induced canonical morphism*

$$\mathfrak{X}^\circ \longrightarrow \mathfrak{X} := \operatorname{Spec} R$$

is an open embedding restricting to $X_0^\circ \to X_0$ modulo I_0. In particular, R is a flat $A[Q]/I$-module, so that $\mathfrak{X}$ is flat over $\operatorname{Spec} A[Q]/I$.

Proof. The proof is completely analogous to the proof of Proposition 3.17. In fact, the only properties used are (1) X_0 is affine, (2) the reductions modulo I_0 of the ϑ_m generate $\Gamma(X_0, \mathcal{O}_{X_0})$ and (3) flatness of $\mathfrak{X}^\circ$ over $A[Q]/I$. □

Example 3.22. We are now in position to finish the discussion of Example 2.17. In this example, the asymptotic monomials of $(B,\mathscr{P})$ are in bijection with integral points of $B\cap\mathbb{Z}^2\setminus\{(0,0)\}$. If $m=(a,b)$ with $a\le 0$ then $\vartheta_m = z^{(a,b,0)} = x^{-a}w^b$ in $R^\partial_{\mathfrak{u}_1}$, while if $a\ge 0$ then $\vartheta_m = z^{(a,b,a)} = y^a w^{b-a}$ in $R^\partial_{\mathfrak{u}_2}$. Taking into account the transport of monomials we see that

$$X=\vartheta_{(-1,0)},\quad Y=\vartheta_{(1,1)},\quad W=\vartheta_{(0,1)}.$$

In fact, say for X, we find that an interior point of σ_2 is the endpoint of two broken lines with asymptotic monomial $(-1,0)$. This yields the expression $(1+w)y^{-1}t$ that we gave for the restriction of X to $\operatorname{Spec} R^\partial_{\mathfrak{u}_2}$.

Moreover, by working in $R^\partial_{\mathfrak{u}_1}$ or in $R^\partial_{\mathfrak{u}_2}$, any other ϑ_m can be written as a polynomial in X,W or in Y,W. Thus by Proposition 3.21 X,Y and W generate $R=\Gamma(\mathfrak{X}^\circ,\mathcal{O}_{\mathfrak{X}^\circ})$, and they provide the description of $\mathfrak{X}^\circ$ as the open subset of $\operatorname{Spec} R$ claimed in Example 2.17.

Example 3.23. In the case of (B,Σ) arising from a Looijenga pair (Y,D) as covered in Examples 1.3,2, 1.11,2, 2.5, 2.15,2 and 3.11 we note (B,Σ) is conical. In particular, since the canonical scattering diagram provides a consistent wall structure, Proposition 3.21 provides a flat deformation of the n-vertex $\mathbb{V}_n$. Note that Proposition 3.21 is a generalization of Theorem 2.26 of [GHK1].

3.5. **The multiplicative structure.** In this section we give an a priori definition of the ring structure on $\bigoplus_m \big(A[Q]/I\big)\cdot\vartheta_m$ turning the map

$$\bigoplus_m \big(A[Q]/I\big)\vartheta_m \longrightarrow \Gamma\big(\mathfrak{X}^\circ, \mathcal{O}_{\mathfrak{X}^\circ}\big)$$

into an isomorphism of $A[Q]/I$-algebras. Our multiplication rule is tropical in the sense that it is purely in terms of broken lines.

Theorem 3.24. *Let $\mathscr{S}$ be a consistent wall structure on the polyhedral pseudomanifold $(B,\mathscr{P})$, and let $\mathfrak{X}^\circ$ be the corresponding flat scheme over $A[Q]/I$ (Proposition 2.16). For asymptotic monomials m_1, m_2 let*

$$\vartheta_{m_1}\cdot\vartheta_{m_2} = \sum_m \alpha_m(m_1,m_2)\cdot\vartheta_m \tag{3.11}$$

be the expansion according to the direct sum decomposition of Theorem 3.19. Thus the sum runs over the asymptotic monomials of $(B,\mathscr{P})$ and $\alpha_m(m_1,m_2)\in A[Q]/I$ is non-zero only for finitely many m.

For an asymptotic monomial m let $\mathfrak{u}$ be an unbounded chamber of $\mathscr{S}$ such that m is an asymptotic monomial on $\mathfrak{u}$. Let $p\in\mathfrak{u}$ be a point that is general for broken lines of asymptotics m_1 and m_2. Then

$$\alpha_m(m_1,m_2) = \sum_{(\beta_1,\beta_2)} a_{\beta_1}a_{\beta_2},$$

where the sum is over all pairs (β_1,β_2) of broken lines with asymptotics m_1, m_2, with endpoint p and such that $\overline{m}_{\beta_1}+\overline{m}_{\beta_2}=m$, viewed as an equation in $\Lambda_{\sigma_\mathfrak{u}}$.

Proof. This proof is a straightforward adaptation from [GHK1], §2.4. To find the coefficient $\alpha_m(m_1,m_2)$ in the stated expansion we look at the coefficients of z^m in $R_\mathfrak{u} = (A[Q]/I)[\Lambda_\sigma]$ of both sides of (3.11). Now the only broken line β with endpoint $p\in\operatorname{Int}\mathfrak{u}$ and with $\overline{m}_\beta = m$ lies entirely in $\mathfrak{u}$ and has no bends. Thus in the local expression of the canonical functions in $R_\mathfrak{u}$ only ϑ_m has a non-zero coefficient of z^m, which is 1. Thus $\alpha_m(m_1,m_2)$ agrees with the coefficient of z^m in the expansion of the left-hand side in $R_\mathfrak{u}$. The statement now follows readily by plugging in the local definition of ϑ_{m_1} and ϑ_{m_2} in terms of broken lines with the respective asymptotics. □

4. The projective case — theta functions

In the case that X_0 is not affine we are going to construct an extension $\mathcal{L}^\circ$ of the ample line bundle on X_0 to $\mathfrak{X}^\circ$, and an $A[Q]/I$-module basis of global sections of powers $(\mathcal{L}^\circ)^{\otimes d}$ for $d\ge 0$. This is done by constructing the total space

$\mathfrak{L}^\circ$ of $(\mathcal{L}^\circ)^{-1}$ as an affine scheme over $\mathfrak{X}^\circ$. The canonical sections of $(\mathcal{L}^\circ)^{\otimes d}$ are then constructed as fibrewise homogeneous canonical functions on $\mathfrak{L}^\circ$ of the kind considered in Section 3. Eventually we can then define the partial completion $\mathfrak{X}$ of $\mathfrak{X}^\circ$ as $\operatorname{Proj}\big(\bigoplus_d \Gamma(\mathfrak{X}^\circ, (\mathcal{L}^\circ)^{\otimes d})\big)$.

On the tropical side the transition from $\mathfrak{X}^\circ$ to $\mathfrak{L}^\circ$ corresponds to taking a truncated cone over $(B, \mathscr{P})$. We begin with an investigation of the cone construction.

4.1. **Conical affine structures.** Let B_0 be an affine manifold (without singularities and not necessarily integral for the moment, see §1.1, and possibly with $\partial B \neq \emptyset$). Thus B_0 is a real manifold of dimension n with an atlas such that the transition functions are affine transformations $T \in \operatorname{Aff}(\mathbb{R}^n) = \operatorname{GL}(n, \mathbb{R}) \ltimes \mathbb{R}^n$. Our notation for affine transformations of a real vector space V is $T = A + b$ with $A \in \operatorname{GL}(V)$, $b \in V$.

Construction 4.1. *(The cone over an affine manifold)* The *cone over* B_0 is the cone of B_0 as a topological space

$$\mathbf{C}B_0 := \big(B_0 \times \mathbb{R}_{\geq 0}\big)/\big(B_0 \times \{0\}\big),$$

endowed with the following affine structure with singularity at the *apex* $O \in \mathbf{C}B_0$, the image of $B_0 \times \{0\}$ in $\mathbf{C}B_0$. For $\psi : U \to \mathbb{R}^n$ an affine chart for B_0, defined on an open set $U \subseteq B_0$ we define the chart

$$\tilde{\psi} : \mathbf{C}U \setminus \{O\} \longrightarrow \mathbb{R}^{n+1}, \quad (x, h) \longmapsto (h \cdot \psi(x), h) \tag{4.1}$$

for $\mathbf{C}B_0$.

We remark that if B_0 is unbounded, it is not really appropriate for the cone to have an apex, but rather the apex should be replaced by an asymptotic version of B_0. This is easier to do when given a polyhedral pseudomanifold, see Definition 4.4. However, the precise nature of the apex will not play a role in the discussion in this subsection. □

Thus if two charts ψ_1, ψ_2 are related by $\psi_2 = A \circ \psi_1 + b$ for $A \in \operatorname{GL}(n, \mathbb{R})$, $b \in \mathbb{R}^n$ then

$$\tilde{\psi}_2 = \tilde{A} \circ \tilde{\psi}_1$$

with $\tilde{A}(x, h) = (Ax + hb, h)$. Intrinsically, if $\mathbb{A}$ is an integral affine space with underlying real vector space V, then the map associating to a pair $(A, b) \in \operatorname{GL}(\mathbb{R}^n) \times \mathbb{R}^n$ the linear transformation $\tilde{A} \in \operatorname{GL}(\mathbb{R}^{n+1})$ generalizes to

$$\operatorname{Aff}(V) \longrightarrow \operatorname{GL}(V \oplus \mathbb{R}), \quad A + b \longmapsto \tilde{A}.$$

We refer to this process as *homogenization* of the affine transformation $A + b$. Clearly, if $A + b \in \operatorname{Aff}(T_x B_0)$ is the affine holonomy along a closed path γ on B_0 starting and ending at x, then $\tilde{A}$ is the affine monodromy of (γ, h) for any $h > 0$.

We think of the cone as standing on the apex and call the second entry h the *height* of $(x,h)\in \mathbf{C}B_0$.

Note that all transition functions of $\mathbf{C}B_0\setminus\{O\}$ are linear. Hence $\mathbf{C}B_0\setminus\{O\}$ is a radiant affine manifold, that is, has vanishing radiance obstruction ([GrSi2], Definition 1.6). Further features are that for $h>0$ the rescaled affine manifold hB_0 with charts $h\psi$ is embedded as the affine submanifold $B_0\times\{h\}$ of constant height. Moreover, for any $x\in B_0$ the ray

$$L_x := \big\{(y,h)\in \mathbf{C}B_0\setminus\{O\}\,\big|\, y=x\big\}$$

is an affine line. However, if $x\neq y$ then L_x and L_y are not parallel as they would be in the product affine manifold $B_0\times\mathbb{R}_{\ge 0}$. To quantify this, we can consider the flat *affine* connection on $\mathbf{C}B_0\setminus\{0\}$ induced by the affine structure on $\mathbf{C}B_0$.[9] Identifying the tangent space of $\mathbf{C}B_0\setminus\{O\}$ at (x,h) with $T_xB_0\oplus\mathbb{R}$ with the second factor the tangent space to L_x, we have the following description of parallel transport with respect to this connection.

Proposition 4.2. *Let*

$$T_\gamma = A+b : T_xB_0 \longrightarrow T_yB_0,$$

be the affine parallel transport for a path γ in B_0 from x to y and let $b\in\Lambda_y$ be the affine displacement vector (in an affine chart, $b=x-y$). Then the linear part of the parallel transport on $\mathbf{C}B_0\setminus\{O\}$ from (x,h_1) to (y,h_2) along a path of the form $t\mapsto(\gamma(t),h(t))$ is given by

$$T_xB_0\oplus\mathbb{R}\longrightarrow T_yB_0\oplus\mathbb{R},\quad (v,\eta)\longmapsto \big(h_2^{-1}(h_1Av+\eta b),\eta\big).$$

Proof. By a straightforward computation the claimed formula is compatible with compositions of paths. Hence we can restrict to the domain of a single chart, and in turn to B_0 an open subset of $\mathbb{R}^n$. Let $x_1,\ldots,x_n$ be the affine coordinates on B_0 thus defined and consider x_i as functions on $\mathbf{C}B_0$ by pull-back via the projection $\mathbf{C}B_0\setminus\{O\}\to B_0$. Affine parallel transport on B_0 in this chart gives $A=\mathrm{id}$ and $b=\sum_i(x_i(x)-x_i(y))\partial_{x_i}$. The x_i together with the height function h define a non-affine coordinate chart on $\mathbf{C}B_0\setminus\{O\}$. Let $w_1,\ldots,w_{n+1}$ be the affine coordinate functions on $\mathbf{C}B_0\setminus\{O\}$ defined by (4.1) for the given chart of B_0. In particular, $\partial_{w_1},\ldots,\partial_{w_{n+1}}$ define a basis of flat vector fields on $\mathbf{C}B_0\setminus\{O\}$. Since $h=w_{n+1}$ and $x_i=w_{n+1}^{-1}w_i=h^{-1}w_i$ we have

$$\partial_{w_i}=h^{-1}\partial_{x_i},\quad \partial_{w_{n+1}}=\partial_h+\sum_{i=1}^n(-h^{-2}w_i)\partial_{x_i}=\partial_h-h^{-1}\sum_{i=1}^n x_i\partial_{x_i}.$$

[9] There is a confusion in the literature about the attributes “linear” versus “affine” for connections. Affine connections in the sense used here take into account the moving of the base point also, see e.g. [KoNi], Chapter III.

Thus $h^{-1}\partial_{x_i}$ and $\partial_h - h^{-1}\sum_i x_i\partial_{x_i}$ are a basis of flat vector fields on $\mathbf{C}B_0 \setminus \{O\}$. Evaluating at (x, h_1) and on (y, h_2) now establishes the claimed formula for the linear part of the parallel transport on $\mathbf{C}B_0 \setminus \{O\}$. □

Remark 4.3. 1) The proposition shows that the parallel transport of the *linear* connection on $\mathbf{C}B_0 \setminus \{O\}$ contains all the information of *affine* parallel transport on B_0. Note also that for a closed loop on $\mathbf{C}B_0 \setminus \{O\}$ affine parallel transport is linear because $\mathbf{C}B_0 \setminus \{O\}$ is radiant.

2) A special case is that $h_1 = h_2 = h$, for example if γ is a closed loop. Then the map reads

$$(v, \eta) \longmapsto \big(Av + h^{-1}\eta b, \eta\big). \tag{4.2}$$

3) If B_0 is integral then also $\mathbf{C}B_0$ is integral, and all of the stated formulas respect the integral structure. But note that the affine embedding $B_0 \times \{h\} \hookrightarrow \mathbf{C}B_0$ is integral only for $h = 1$.

4.2. **The cone over a polyhedral pseudomanifold.** Let us now assume that $B_0 = B \setminus \Delta$ for a polyhedral pseudomanifold $(B, \mathscr{P})$. Recall from (2.1) the definition of $\mathbf{C}\sigma$ for σ a (possibly unbounded) polyhedron. In particular, if $\sigma \subseteq \mathbb{R}^n$, then the intersection of $\mathbf{C}\sigma$ with $\mathbb{R}^n \times \{0\}$ is the asymptotic cone of σ. If $(\tau_1 \to \tau_2) \in \hom(\mathscr{P})$ identifies τ_1 with a face of τ_2 then taking cones yields an identification of $\mathbf{C}\tau_1$ with a face of $\mathbf{C}\tau_2$.

Definition 4.4. The *cone over the polyhedral pseudomanifold* $(B, \mathscr{P})$ is the topological space

$$\mathbf{C}B = \varinjlim_{\sigma \in \mathscr{P}} \mathbf{C}\sigma$$

with polyhedral decomposition $\mathbf{C}\mathscr{P} := \big\{\mathbf{C}\tau \,\big|\, \tau \in \mathscr{P}\big\}$ and affine structure on $\mathbf{C}B_0 \subseteq \mathbf{C}B \setminus \mathbf{C}\Delta$ defined in Construction 4.1.

Note that the affine structure on $\mathbf{C}B_0$ extends uniquely to the closure in (2.1) in a way compatible with the inclusion of faces. Thus $(\mathbf{C}B, \mathbf{C}\mathscr{P})$ is a polyhedral affine pseudomanifold as defined in Construction 1.1.

Clearly, $(\mathbf{C}B, \mathbf{C}\mathscr{P})$ is conical (Definition 3.20). Note that the projection to the second factor $\mathbb{R}$ in (2.1) defines a global affine function $h : \mathbf{C}B \to \mathbb{R}$, the height, and $h^{-1}(0)$ is the union of the asymptotic cones of $\sigma \in \mathscr{P}$. Normalizing by the height defines a deformation retraction

$$\mathbf{C}B \setminus h^{-1}(0) \to B \times \{1\}$$

with preimage of a subset $A \subseteq B = B \times \{1\}$ the punctured cone $\mathbf{C}A \setminus h^{-1}(0)$ over A.

Our next objective is to lift a wall structure $\mathscr{S}$ on $(B,\mathscr{P})$ to $(\mathbf{C}B,\mathbf{C}\mathscr{P})$. Note first that a Q^{gp}-valued MPA function φ on B induces the MPA function $\mathbf{C}\varphi$ on $\mathbf{C}B$ with kinks

$$\kappa_{\mathbf{C}\underline{\rho}}(\mathbf{C}\varphi) := \kappa_{\underline{\rho}}(\varphi).$$

This definition makes sense because the connected components of $\mathbf{C}\rho\setminus\mathbf{C}\Delta$ are cones over the connected components of $\rho\setminus\Delta$. The restriction of a local representative of $\mathbf{C}\varphi$ to $B = B\times\{1\}$ is a local representative of φ. In fact, this is a nontrivial statement only at general points of a codimension one cell $\mathbf{C}\underline{\rho}$, $\underline{\rho}\in\tilde{\mathscr{P}}^{[n-1]}$. By the definition of $\underline{\rho}$ there is a vertex $v\in\underline{\rho}$. Then $\Lambda_{\mathbf{C}\rho} = (\Lambda_\rho\times\{0\})\oplus(\mathbb{Z}\cdot(v,1))$. With this description of $\Lambda_{\mathbf{C}\rho}$, if $x\in\operatorname{Int}\underline{\rho}$ and $\xi\in\Lambda_x$ generates $\Lambda_{B,x}/\Lambda_\rho$ then $(\xi,0)$ generates $\Lambda_{\mathbf{C}B,x}/\Lambda_{\mathbf{C}\rho}$. The statement now follows from the definition of the kink of an MPA function from a local representative (Definition 1.6).

For the monomials on $\mathbf{C}B_0$ use the integral affine embedding $B\times\{1\}\to\mathbf{C}B$ and parallel transport along rays emanating from the apex $O\in\mathbf{C}B$ to lift a monomial at $x\in B$ to a monomial at any point on $\mathbf{C}x = \{x\}\times\mathbb{R}_{\ge 0}\subseteq\mathbf{C}B$. By abuse of notation we interpret a monomial $m\in\mathcal{P}_x$ at a point $x\in B\setminus\Delta$ (a monomial on B_0) also as a monomial on $\mathbf{C}B$ at any point $(x,h)\in\mathbf{C}x$.

The lifting of a wall $\mathfrak{p}$ of codimension zero shows a certain subtlety that we now explain. Let $\sigma\in\mathscr{P}_{\max}$ be the maximal cell containing $\mathfrak{p}$ and let $n\in\check\Lambda_\sigma$ generate $\Lambda_{\mathfrak{p}}^\perp\subseteq\check\Lambda_\sigma$. Projection to the last component (the height) induces the map of lattices

$$\Lambda_{\mathbf{C}\mathfrak{p}}\longrightarrow\mathbb{Z}.$$

If this map is surjective then there exists $b\in\mathbb{N}$ with $(n,-b)$ a generator of $\Lambda_{\mathbf{C}\mathfrak{p}}^\perp\subseteq\check\Lambda_{\mathbf{C}\sigma}$. In fact, if $(m,1)\in\Lambda_{\mathbf{C}\mathfrak{p}}$ is a lift of $1\in\mathbb{Z}$, then $\Lambda_{\mathbf{C}\mathfrak{p}} = \Lambda_{\mathfrak{p}}\times\{0\}\oplus\mathbb{Z}\cdot(m,1)$; in this case $(n,-b)$ with $b:=\langle n,m\rangle$ generates $\Lambda_{\mathbf{C}\mathfrak{p}}^\perp$. In general, the image of $\Lambda_{\mathbf{C}\mathfrak{p}}\to\mathbb{Z}$ is only a subgroup of $\mathbb{Z}$, hence of the form $a\cdot\mathbb{Z}$ for some $a\in\mathbb{N}$. Let $(m,a)\in\Lambda_{\mathbf{C}\mathfrak{p}}$ be a lift. Then $\Lambda_{\mathbf{C}\mathfrak{p}} = \Lambda_{\mathfrak{p}}\times\{0\}\oplus\mathbb{Z}\cdot(m,a)$ and

$$\Lambda_{\mathbf{C}\mathfrak{p}}^\perp = \mathbb{Z}\cdot(an,-b)$$

with $b = \langle n,m\rangle$.

Definition 4.5. For a polyhedral subset $\mathfrak{a}\subseteq B$ the index $a\in\mathbb{N}$ of the image of the projection $\Lambda_{\mathbf{C}\mathfrak{a}}\to\mathbb{Z}$ to the height is called the *index of* $\mathbf{C}\mathfrak{a}$.

Thus if we want to lift the wall in such a way that the attached automorphism is compatible with the automorphism attached to $\mathfrak{p}$ we need to take an a-th root of $f_{\mathfrak{p}}$ for a the index of $\mathbf{C}\mathfrak{p}$. Such a root exists uniquely by the following elementary lemma whose proof is left to the reader.

Lemma 4.6. *Let R be a ring containing $\mathbb{Q}$ and $I_0\subseteq R$ a nilpotent ideal. Then for any $f\in 1+I_0$ and $a\in\mathbb{N}\setminus\{0\}$ there exists a unique $g\in 1+I_0$ with $g^a = f$.* □

Definition 4.7. The *cone of a wall* $(\mathfrak{p}, f_\mathfrak{p})$ on the polyhedral pseudomanifold $(B, \mathscr{P})$ is the wall on $(\mathbf{C}B, \mathbf{C}\mathscr{P})$ with underlying set $\mathbf{C}\mathfrak{p}$ and function $f_{\mathbf{C}\mathfrak{p}} := f_\mathfrak{p}^{1/a}$, with the monomials on B canonically interpreted as monomials on $\mathbf{C}B$ as explained. Here a is the index of $\mathbf{C}\mathfrak{p}$ (Definition 4.5) and $f_\mathfrak{p}^{1/a}$ is the a-th root of $f_\mathfrak{p}$ according to Lemma 4.6.

Taking cones of the elements of a wall structure $\mathscr{S}$ on $(B, \mathscr{P})$ defines the wall structure $\mathbf{C}\mathscr{S}$ on $(\mathbf{C}B, \mathbf{C}\mathscr{P})$. Technically, under certain circumstances, this will not satisfy the definition of wall structure. Indeed, if $\mathscr{S}$ has a chamber $\mathfrak{u}$ whose asymptotic cone $\mathfrak{u}_\infty$ is $n = \dim B$-dimensional, and if in addition $\mathfrak{u}$ intersects ∂B in a set of dimension $n-1$, then $\mathbf{C}\mathfrak{u}$ will intersect two different n-dimensional cells of $\partial\mathbf{C}B$ in n-dimensional sets. This violates condition (2)(c) of Definition 2.11. This can be rectified by adding some walls to $\mathbf{C}\mathscr{S}$ which have attached function 1. Since such walls do not affect anything, we will ignore this technical issue.

Remark 4.8. Note that there are no roots involved in codimension one walls since they are contained in facets of the adjacent maximal cells, which contain integral points, and hence they have index one. Slab functions are not of the form covered by Lemma 4.6 and may not have roots.

Proposition 4.9. *If the wall structure $\mathscr{S}$ on $(B, \mathscr{P})$ is consistent (in codimension k) then so is the lifted wall structure $\mathbf{C}\mathscr{S}$ on $(\mathbf{C}B, \mathbf{C}\mathscr{P})$.*

Proof. (Consistency in codimension zero.) Let $\mathfrak{j} \subseteq B$ be a joint for $\mathscr{S}$ of codimension zero, contained in some $\sigma \in \mathscr{P}_{\max}$. Label the adjacent walls $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ cyclically and let $\theta_{\mathfrak{p}_1}, \ldots, \theta_{\mathfrak{p}_r}$ be the associated automorphisms of R_σ. Then consistency of $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ reads

$$\theta_{\mathfrak{p}_r} \circ \cdots \circ \theta_{\mathfrak{p}_1} = \mathrm{id}\,.$$

With the identification of monomials at $x \in \operatorname{Int}\sigma$ with monomials on $\mathbf{C}x$ this equation readily implies the claimed consistency

$$\big(\theta_{\mathbf{C}\mathfrak{p}_r} \circ \cdots \circ \theta_{\mathbf{C}\mathfrak{p}_1}\big)(z^m) = z^m \tag{4.3}$$

for all monomials m coming from B. Indeed, if m is a monomial defined on a wall $\mathfrak{p}$ of $\mathscr{S}$ with $\mathbf{C}\mathfrak{p}$ of index a and $\theta_\mathfrak{p}(z^m) = f_\mathfrak{p}^{\langle n_\mathfrak{p}, \overline{m}\rangle} \cdot z^m$, then viewing m as a monomial on $\mathbf{C}B$ it holds

$$\theta_{\mathbf{C}\mathfrak{p}}(z^m) = f_{\mathbf{C}\mathfrak{p}}^{a\langle n_\mathfrak{p}, \overline{m}\rangle} \cdot z^m = f_\mathfrak{p}^{\langle n_\mathfrak{p}, \overline{m}\rangle} \cdot z^m.$$

Since $\Lambda_{\mathbf{C}\sigma} = \Lambda_\sigma \times \{0\} \oplus \mathbb{Z} \cdot (0,1)$ it remains to show (4.3) for $m = (0,1)$. Here $(0,1) \in \Lambda_\sigma \oplus \mathbb{Z}$ is viewed as a monomial on $\operatorname{Int}\mathbf{C}\sigma$ with vanishing Q-component via (1.10). Since $\big(\theta_{\mathbf{C}\mathfrak{p}_1} \circ \ldots \circ \theta_{\mathbf{C}\mathfrak{p}_r}\big)(z^m) = (1+h) \cdot z^m$ with $h \in I_0 \cdot R_\sigma$ and in view of the uniqueness statement in Lemma 4.6, it suffices to prove (4.3) for any power of $z^{(0,1)}$. Let $(m,a) \in \Lambda_{\mathbf{C}\mathfrak{j}}$ be such that $a \in \mathbb{N}$ is the index of $\mathbf{C}\mathfrak{j}$. Now $(0,a) = (m,a) - (m,0)$ with $m \in \Lambda_\sigma$, and (4.3) already holds for $z^{(m,0)}$, while $z^{(m,a)}$ is left invariant by any of the $\theta_{\mathbf{C}\mathfrak{p}_i}$. Hence (4.3) holds for all monomials m on $\mathbf{C}\sigma$.

(Consistency in codimension one.) Let $\mathfrak{j}$ be a codimension one joint and $\rho \in \mathscr{P}^{[n-1]}$ the codimension one cell containing $\mathfrak{j}$. As in Definition 2.14 let $\mathfrak{b}_1, \mathfrak{b}_2 \in \mathscr{S}$ be the slabs adjacent to $\mathfrak{j}$ and θ, θ' the automorphisms of R_σ, $R_{\sigma'}$ induced by passing through the walls containing $\mathfrak{j}$ in the correct order. Here θ and θ' collect the walls in the two maximal cells σ, σ' containing ρ, respectively. Let $\chi_{\mathfrak{b}_i,\sigma/\sigma'}$: $R_{\mathfrak{b}_i} \to R_{\sigma/\sigma'}$ be the natural ring homomorphisms. Consistency of $\mathscr{S}$ around $\mathfrak{j}$ says

$$(\theta\times\theta')\big((\chi_{\mathfrak{b}_1,\sigma},\chi_{\mathfrak{b}_1,\sigma'})(R_{\mathfrak{b}_1})\big) = (\chi_{\mathfrak{b}_2,\sigma},\chi_{\mathfrak{b}_2,\sigma'})(R_{\mathfrak{b}_2}).$$

The argument now runs analogously to the codimension zero case. Using a chart around $x \in \mathfrak{b}_1$ with 0 in the affine span of the image of $\mathfrak{b}_1$ shows that

$$\Lambda_{\mathbf{C}\rho} = \Lambda_\rho \oplus \mathbb{Z}\cdot(0,1).$$

In particular, for $x \in \operatorname{Int}\mathfrak{b}$ the generator of $\Lambda_{\mathbf{C}B,x}/\Lambda_{\mathbf{C}\rho}$ leading to the monomials Z_+,Z_- can be chosen to lie in $\Lambda_\sigma \oplus 0$. With this identification and choice we have

$$R_{\mathbf{C}\mathfrak{b}_i} = \big(A[Q]/I\big)[\Lambda_\rho][z^{(0,1)}, Z_+, Z_-]/(Z_+Z_- - f_{\mathfrak{b}_i} z^{\kappa_{\underline{\rho}_i}}),$$

with $\underline{\rho}_i \supseteq \mathfrak{b}_i$ the $(n-1)$-cell of the barycentric subdivision containing $\mathfrak{b}_i$. Let $\tilde\theta$, $\tilde\theta'$ be the automorphisms of $R_{\mathbf{C}\sigma}$, $R_{\mathbf{C}\sigma'}$ induced by crossing the walls containing $\mathbf{C}\mathfrak{j}$ on $\mathbf{C}B$. Writing $\tilde\chi_{\mathfrak{b}_i,\sigma/\sigma'} : R_{\mathbf{C}\mathfrak{b}_i} \to R_{\mathbf{C}\sigma/\mathbf{C}\sigma'}$ for the natural localization homomorphisms, the equation

$$(\tilde\theta\times\tilde\theta')\big((\tilde\chi_{\mathfrak{b}_1,\sigma},\tilde\chi_{\mathfrak{b}_1,\sigma'})(R_{\mathbf{C}\mathfrak{b}_1})\big) = (\tilde\chi_{\mathfrak{b}_2,\sigma},\tilde\chi_{\mathfrak{b}_2,\sigma'})(R_{\mathbf{C}\mathfrak{b}_2}) \tag{4.4}$$

for consistency around $\mathbf{C}\mathfrak{j}$ already holds for monomials lifted from B. In fact, for $m \in \Lambda_\sigma$ we have seen in the treatment of consistency in codimension zero that $\tilde\theta(z^m) = \theta(z^m)$ (with the usual abuse of notation of interpreting monomials on B as monomials on $\mathbf{C}B$), and similarly for σ' and $\tilde\theta'$. Since Z_+, Z_- are monomials lifted from B, both for $R_{\mathbf{C}\mathfrak{b}_1}$ and $R_{\mathbf{C}\mathfrak{b}_2}$, the equality (4.4) holds for any monomial lifted from B.

It remains to treat $z^{(0,1)} \in R_{\mathbf{C}\mathfrak{b}_1}$. Let $a \in \mathbb{N}\setminus\{0\}$ be the index of $\mathbf{C}\mathfrak{j}$ and let $m \in \Lambda_\rho$ be such that $(m,a) \in \Lambda_{\mathbf{C}\mathfrak{j}}$. Then (m,a) is tangent to each wall containing $\mathbf{C}\mathfrak{j}$ and hence

$$\begin{aligned}(\tilde\theta\times\tilde\theta')\big((\tilde\chi_{\mathfrak{b}_1,\sigma},\tilde\chi_{\mathfrak{b}_1,\sigma'})(z^{(m,a)})\big) &= \big(\tilde\theta,\tilde\theta'\big)(z^{(m,a)})\\ &= \big(z^{(m,a)}, z^{(m,a)}\big) = (\tilde\chi_{\mathfrak{b}_2,\sigma},\tilde\chi_{\mathfrak{b}_2,\sigma'})(z^{(m,a)}).\end{aligned} \tag{4.5}$$

Moreover, $(m,0)$ is a monomial lifted from B, and m is invariant under monodromy around $\mathfrak{j}$ for $m \in \Lambda_\rho$. Thus by consistency on B there exists $h \in R_{\mathfrak{b}_2}$ with

$$(\theta,\theta')(z^m) = (\chi_{\mathfrak{b}_2,\sigma},\chi_{\mathfrak{b}_2,\sigma'})(h) \tag{4.6}$$

and h is congruent to z^m modulo I_0. Since $\chi_{\mathfrak{b}_2,\sigma/\sigma'}(h)$ is thus obtained from z^m by wall crossing there exists $f \in 1 + I_0 \cdot R_{\mathfrak{b}_2}$ with $h = f \cdot z^m$. Hence it holds $\big(\tilde\theta,\tilde\theta'\big)(z^{(m,0)}) = (\tilde\chi_{\mathfrak{b}_2,\sigma},\tilde\chi_{\mathfrak{b}_2,\sigma'})(f \cdot z^{(m,0)})$. Together with (4.5) this shows $\big(\tilde\theta,\tilde\theta'\big)(z^{(0,a)}) = (\tilde\chi_{\mathfrak{b}_2,\sigma},\tilde\chi_{\mathfrak{b}_2,\sigma'})(f^{-1} \cdot z^{(0,a)})$. Taking roots according to Lemma 4.6 then yields

$$\big(\tilde\theta,\tilde\theta'\big)(z^{(0,1)}) = (\tilde\chi_{\mathfrak{b}_2,\sigma},\tilde\chi_{\mathfrak{b}_2,\sigma'})(f^{-1/a} \cdot z^{(0,1)}),$$

establishing (4.4) for the remaining generator of $R_{\mathfrak{b}_1}$.

(Consistency in codimension two.) Let $\mathfrak{j}$ be a codimension two joint, and let $\tau \in \mathscr{P}^{[n-2]}$ be the minimal cell containing $\mathfrak{j}$. In contrast to the previous cases of codimension zero and one, the index of $\mathbf{C}\mathfrak{j}$ is always one. Indeed, since τ has integral points, $\mathbf{C}\tau$ has index one, and

$$\Lambda_{\mathbf{C}\mathfrak{j}} = \Lambda_{\mathbf{C}\tau},$$

because $\operatorname{Int}\mathfrak{j}$ is an open subset of τ. Thus in a chart for any $\sigma \in \mathscr{P}_{\max}$ containing $\mathfrak{j}$ and centered at an integral point of τ we have the decomposition

$$\Lambda_{\mathbf{C}\sigma} = \big(\Lambda_\sigma \times \{0\}\big) \oplus \mathbb{Z}\cdot(0,1).$$

Now consistency around $\mathfrak{j}$ means that the functions $\vartheta^{\mathfrak{j}}_m(p)$ do not depend on the choice of general point $p \in B_{\mathfrak{j}}$ (Definition 3.9). As in codimension zero and one this statement is immediate for monomials in $\Lambda_\sigma \times \{0\}$, that is, for monomials lifted from B. On the other hand, a monomial tangent to $\mathfrak{j}$ is left invariant by any of the ring homomorphisms changing chambers. In particular, $\vartheta^{\mathfrak{j}}_{(0,1)}(p) = z^{(0,1)}$ for any p in the interior of a chamber $\mathfrak{u}$ of $\mathscr{S}_{\mathfrak{j}}$. This proves consistency around $\mathfrak{j}$. □

For later use we also express here the asymptotic monomials (Definition 3.1) of $\mathbf{C}B$ in terms of the geometry of B. First note that the projection to the height maps any tangent vector $\overline{m}$ of a monomial m on $\mathbf{C}B$ to an integral tangent vector on $\mathbb{R}$. We call this integer the *degree of* m, written $\deg m$. If m is an asymptotic monomial of $\mathbf{C}B$ then $\deg m \in \mathbb{N}$.

Proposition 4.10. *The set of asymptotic monomials on $\mathbf{C}B$ of degree $d > 0$ are in canonical bijection with the set $B\big(\frac{1}{d}\mathbb{Z}\big)$ of $1/d$-integral points of B. The set of asymptotic monomials on $\mathbf{C}B$ of degree $d = 0$ are in canonical bijection with the set of asymptotic monomials of B.*

Proof. For an integral polyhedron $\sigma \subseteq \Lambda_{\mathbb{R}}$ an asymptotic monomial on $\mathbf{C}\sigma$ is just an element of $\mathbf{C}\sigma \cap (\Lambda \times \mathbb{Z})$, that is, an integral point in the cone. If (m, d) is such a point then d is the degree of the asymptotic monomial and $m \in d \cdot \sigma \cap \Lambda$. For $d > 0$ this means $\frac{1}{d}m \in \sigma \cap \left(\frac{1}{d}\Lambda\right)$, giving a $1/d$-integral point of σ; for $d = 0$ we have an asymptotic monomial of $\mathbf{C}\sigma \cap \left(\Lambda_{\mathbb{R}} \times \{0\}\right)$, that is, an asymptotic monomial of the asymptotic cone σ_∞ of σ.

The general statement follows from the statement for an individual cell since the identification of asymptotic monomials on faces is compatible with the stated identification of asymptotic monomials on $\mathbf{C}\sigma$. □

4.3. **Theta functions and the Main Theorem.** Starting from a consistent wall structure $\mathscr{S}$ on the polyhedral pseudomanifold $(B, \mathscr{P})$, we have now arrived at a consistent wall structure $\mathbf{C}\mathscr{S}$ on the cone $(\mathbf{C}B, \mathbf{C}\mathscr{P})$ of $(B, \mathscr{P})$, see Definition 4.7 and Proposition 4.9. Then $\mathscr{S}$ and $\mathbf{C}\mathscr{S}$ lead to the schemes $\mathfrak{X}^\circ$ and $\mathfrak{Y}^\circ := \mathfrak{X}^\circ_{\mathbf{C}\mathscr{S}}$, respectively. We can then construct $\mathfrak{W} := \operatorname{Spec}\Gamma(\mathfrak{X}^\circ, \mathcal{O}_{\mathfrak{X}^\circ})$ and $\mathfrak{Y} := \operatorname{Spec}\Gamma(\mathfrak{Y}^\circ, \mathcal{O}_{\mathfrak{Y}^\circ})$. Each will be flat over $A[Q]/I$, with $\mathfrak{Y}^\circ$ an open subset of the affine scheme $\mathfrak{Y}$. The object of the present subsection is the construction of a similarly canonical open embedding $\mathfrak{X}^\circ \hookrightarrow \mathfrak{X}$, now with $\mathfrak{X}$ projective over $\mathfrak{W}$. This will be done by relating $\mathfrak{Y}$ to the total space of $\mathcal{O}_{\mathfrak{X}}(-1)$, the dual of an ample invertible sheaf on $\mathfrak{X}$ coming naturally with the construction.

The first step in establishing this picture is the construction of the total space $\mathfrak{L}^\circ$ of a line bundle over $\mathfrak{X}^\circ$. The sheaf of sections of $\mathfrak{L}^\circ$ will be identified with the restriction to $\mathfrak{X}^\circ$ of $\mathcal{O}_{\mathfrak{X}}(-1)$.

Construction 4.11. *(The truncated cone $\overline{\mathbf{C}}B$ and the associated schemes $\mathfrak{L}^{o,\times} \subseteq \mathfrak{Y}^\circ \subseteq \mathfrak{L}^\circ$.)* Let $(B, \mathscr{P})$ be a polyhedral pseudomanifold. The *truncated cone* $(\overline{\mathbf{C}}B, \overline{\mathbf{C}}\mathscr{P})$ over $(B, \mathscr{P})$ is the polyhedral pseudomanifold with underlying topological space

$$\overline{\mathbf{C}}B := \left\{(x, h) \in \mathbf{C}B \,\middle|\, h \geq 1\right\},$$

endowed with the induced affine structure and induced polyhedral decomposition with cells $\overline{\mathbf{C}}\sigma := \left\{(x, h) \in \mathbf{C}\sigma \,\middle|\, h \geq 1\right\}$, $\sigma \in \mathscr{P}$. Clearly, the boundary of $\overline{\mathbf{C}}B$ decomposes into two parts, one coming from ∂B, one from the truncation:

$$\partial\big(\overline{\mathbf{C}}B\big) = \overline{\mathbf{C}}(\partial B) \cup (B \times \{1\}).$$

If $\mathscr{S}$ is a (consistent) wall structure on $(B, \mathscr{P})$ the wall structure $\mathbf{C}\mathscr{S}$ restricts to a (consistent) wall structure $\overline{\mathbf{C}}\mathscr{S}$ on the truncated cone $(\overline{\mathbf{C}}B, \overline{\mathbf{C}}\mathscr{P})$ (subject to the same caveat of Definition 4.7 of perhaps needing to add trivial walls). Indeed, the only thing to check is consistency of joints introduced by the truncation. These are either of the form $\rho \times \{1\}$ where ρ is an $(n-1)$-dimensional cell in ∂B or $\mathfrak{p} \times \{1\}$ where $\mathfrak{p}$ is a wall in $\mathscr{S}$. However, there are no walls of $\overline{\mathbf{C}}\mathscr{S}$

containing joints of the first sort, and hence consistency follows trivially from Proposition 3.13. For joints of the second sort, there is no consistency condition if $\mathfrak{p}$ is a codimension zero wall. If $\mathfrak{p}$ is a slab, consistency again follows from Proposition 3.13, this time using the fact that all exponents m appearing in $f_{\mathbf{C}\mathfrak{p}}$ have $\overline{m}$ tangent to $B \times \{1\}$. Thus in the consistent case we obtain from $\mathbf{C}\mathscr{S}$ and $\overline{\mathbf{C}}\mathscr{S}$ two flat $A[Q]/I$-schemes $\mathfrak{Y}^\circ$ and $\mathfrak{L}^\circ$. Both schemes are covered by spectra of rings with a $\mathbb{Z}$-grading defined by the degree of monomials, introduced in the text before Proposition 4.10, and the gluings respect the grading. In particular, $\mathfrak{L}^\circ$ and $\mathfrak{Y}^\circ$ come with a $\mathbb{G}_m$-action.

Note that $\mathfrak{L}^\circ$ contains one stratum for every maximal cell $\sigma \in \mathscr{P}$ induced by the cell of the lower boundary $\sigma \times \{1\} \subseteq B \times \{1\}$. On the other hand, if the asymptotic cone σ_∞ of σ has dimension n, then σ_∞ is an n-cell of the lower boundary of $\mathbf{C}B$, and hence there is a stratum of $\mathfrak{Y}^\circ$ indexed by $\sigma_\infty \times \{0\}$. Furthermore, if $\mathfrak{u}$ is a chamber of $\mathscr{S}$ contained in σ with $\mathfrak{u}_\infty$ n-dimensional, then the rings $R^\partial_{\overline{\mathbf{C}}\mathfrak{u}}$ contributing to $\mathfrak{L}^\circ$ and $R^\partial_{\mathbf{C}\mathfrak{u}}$ contributing to $\mathfrak{Y}^\circ$ coincide. In particular, $\mathfrak{Y}^\circ$ is thus a subscheme of $\mathfrak{L}^\circ$.

Let $\mathfrak{L}^{\circ,\times} \subseteq \mathfrak{L}^\circ$ be the open subscheme obtained by deleting the codimension one strata of $\mathfrak{L}^\circ$ corresponding to the lower boundary cells $B \times \{1\} \subseteq \partial(\overline{\mathbf{C}}B)$. This is obtained by gluing together only those charts of the form $\operatorname{Spec} R_\mathfrak{u}$ for any $\mathfrak{u}$, $\operatorname{Spec} R^\partial_\mathfrak{u}$ for those $\mathfrak{u}$ intersecting $\partial(\overline{\mathbf{C}}B) \setminus B \times \{1\}$ in a codimension one set, and $R_\mathfrak{b}$ for $\mathfrak{b}$ a slab. Note that the same set of rings appears in the description of $\mathfrak{Y}^\circ$, and hence $\mathfrak{L}^{\circ,\times} \subseteq \mathfrak{Y}^\circ$ also (and in fact we have equality provided that all cells of $\mathscr{P}$ have asymptotic cone of dimension less than n).

For the rings used for constructing $\mathfrak{L}^\circ$, $\mathfrak{L}^{\circ,\times}$, or $\mathfrak{Y}^\circ$, each subring of elements of degree zero can be identified with one of the rings in the construction of $\mathfrak{X}^\circ$, with each ring for $\mathfrak{X}^\circ$ occurring. Hence $\mathfrak{L}^\circ$, $\mathfrak{L}^{\circ,\times}$ and $\mathfrak{Y}^\circ$ come with a $\mathbb{G}_m$-invariant surjection to $\mathfrak{X}^\circ$.

We claim that $\mathfrak{L}^{\circ,\times}$ has naturally the structure of the total space of a $\mathbb{G}_m$-torsor over $\mathfrak{X}^\circ$, that is, a line bundle minus the zero section. Moreover, we have $\mathfrak{L}^{\circ,\times} \subseteq \mathfrak{Y}^\circ \subseteq \mathfrak{L}^\circ$, with the inclusion of $\mathfrak{L}^{\circ,\times} \subseteq \mathfrak{L}^\circ$ partially compactifying this $\mathbb{G}_m$-torsor by filling in the zero section over the complement of the codimension one strata in $\mathfrak{X}^\circ$.

Local trivializations of $\mathfrak{L}^{\circ,\times}$ are given as follows. For $\rho \in \mathscr{P}^{[n-1]}$, $\rho \not\subseteq \partial B$, any choice of integral point $v \in \rho$ induces an isomorphism $\Lambda_{\mathbf{C}\rho} = \big(\Lambda_\rho \times \{0\}\big) \oplus \mathbb{Z}\cdot(v,1)$. Hence in view of (2.17), for any slab $\mathfrak{b} \in \mathscr{S}$ contained in ρ the choice of an integral point $v \in \rho$ induces an isomorphism of $R_\mathfrak{b}$-algebras

$$R_{\mathbf{C}\mathfrak{b}} \xrightarrow{\simeq} R_\mathfrak{b}[u, u^{-1}],$$

identifying $z^{(v,1)}$ with u. This induces a local trivialization

$$\mathrm{Spec}(R_{\mathbf{C}\mathfrak{b}}) \simeq \mathrm{Spec}(R_{\mathfrak{b}}) \times \mathbb{G}_m.$$

Here $\mathbb{G}_m = \mathrm{Spec}\,\big(\mathbb{Z}[u,u^{-1}]\big)$ and the product is taken over $\mathbb{Z}$. A different choice of integral point leads to the multiplication of u by some z^m with $m \in \Lambda_\rho$, an invertible homomorphism of $R_{\mathfrak{b}}$-algebras. Moreover, any crossing of codimension one joint from $\mathfrak{b}$ to $\mathfrak{b}'$ leads to the multiplication of u by an invertible element in $R_{\mathfrak{b}'}$ and is otherwise compatible with the isomorphism of rings $R_{\mathfrak{b}} \to R_{\mathfrak{b}'}$. Similarly, any integral point on a maximal cell σ induces a local trivialization $\mathrm{Spec}(R_{\mathbf{C}\mathfrak{u}}) \simeq \mathrm{Spec}(R_{\mathfrak{u}}) \times \mathbb{G}_m$ for chambers $\mathfrak{u} \subseteq \sigma$, and wall crossings are again homogeneous of degree zero. This shows that $\mathfrak{L}^{\circ,\times}$ comes with the structure of a $\mathbb{G}_m$-torsor over $\mathfrak{X}^\circ$.

The construction of $\mathfrak{L}^\circ$ only adds $\mathrm{Spec}(R^{\partial}_{\tilde{\mathfrak{u}}})$ for $\tilde{\mathfrak{u}}$ a chamber of $\overline{\mathbf{C}}\mathscr{S}$ that intersects the lower boundary $B \times \{1\} \subseteq \overline{\mathbf{C}}B$ in $\mathfrak{u} \times \{1\}$, where $\mathfrak{u}$ is a chamber of $\mathscr{S}$. Then $R_{\tilde{\mathfrak{u}}} \subseteq R^{\partial}_{\tilde{\mathfrak{u}}}$ leads to the partial $\mathbb{G}_m$-equivariant compactification

$$\mathrm{Spec}(R_{\mathfrak{u}}) \times \mathbb{G}_m \subseteq \mathrm{Spec}(R_{\mathfrak{u}}) \times \mathbb{A}^1.$$

This process adds the zero-section of a line bundle over the complement of the codimension one strata in $\mathfrak{X}^\circ$, as claimed.

We are now in the position to prove one of the main results of this paper.

Theorem 4.12. *Let $\mathscr{S}$ be a consistent wall structure on the polyhedral pseudomanifold $(B,\mathscr{P})$. Denote by $\mathbf{C}\mathscr{S}$ the induced consistent wall structure*[10] *on $(\mathbf{C}B, \mathbf{C}\mathscr{P})$. Let $\mathfrak{X}^\circ$, $\mathfrak{Y}^\circ$ be the associated flat $A[Q]/I$-schemes according to Proposition 2.16 for $\mathscr{S}$ and $\mathbf{C}\mathscr{S}$, respectively. Let*

$$R_\infty := \Gamma\big(\mathfrak{X}^\circ, \mathcal{O}_{\mathfrak{X}^\circ}\big), \quad S := \Gamma\big(\mathfrak{Y}^\circ, \mathcal{O}_{\mathfrak{Y}^\circ}\big)$$

be the $A[Q]/I$-algebras with canonical $A[Q]/I$-module basis of sections ϑ_m constructed in Theorem 3.19. Here m runs through the set of asymptotic monomials on B for R_∞ and on $\mathbf{C}B$ (cf. Proposition 4.10) for S, respectively.

Then the following holds.

(a) *The affine schemes $\mathfrak{W} := \mathrm{Spec}\, R_\infty$ and $\mathfrak{Y} := \mathrm{Spec}\, S$ are flat over $A[Q]/I$.*
(b) *The ring S is a $\mathbb{Z}$-graded R_∞-algebra, with the grading given by* $\deg \vartheta_m := \deg m$. *Also, $S_0 = R_\infty$, where S_0 denotes the degree 0 part of S.*

[10]We assume here $\mathbb{Q} \subseteq A$ to assure the existence of the roots of the wall functions $f_{\mathfrak{p}}$ required in Definition 4.7. In some other cases one can derive the existence of $\mathbf{C}\mathscr{S}$ by a priori methods independently of this assumption. For example, for locally rigid singularities one may run the inductive construction from [GrSi4] directly on $\mathbf{C}B$.

(c) *The scheme* $\mathfrak{X}^\circ$ *embeds canonically as an open dense subscheme into* $\mathfrak{X} := \mathrm{Proj}(S)$*, and* $\mathfrak{X}$ *is flat over* $A[Q]/I$*. Moreover,* $\mathfrak{X}$ *is the unique flat extension of* X_0 *from* $A[Q]/I_0$ *to* $A[Q]/I$ *containing* $\mathfrak{X}^\circ$ *as an open subscheme and proper over* $\mathfrak{W}$*.*

(d) *Denote by* $\mathfrak{L} \to \mathfrak{X}$ *the line bundle with sheaf of sections* $\mathcal{O}_{\mathrm{Proj}(S)}(-1)$*. Then there is a canonical isomorphism* $\Gamma(\mathfrak{L}, \mathcal{O}_{\mathfrak{L}}) \simeq S$ *that induces a morphism* $\mathfrak{L} \to \mathfrak{Y}$ *contracting the zero-section of* $\mathfrak{L}$ *to the fixed locus of the* $\mathbb{G}_m$*-action on* $\mathfrak{Y}$ *defined by the grading of* S*. In particular,* $S = \bigoplus_{d\in\mathbb{N}} \Gamma\big(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}(d)\big)$ *is the homogeneous coordinate ring of* $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}(1))$*.*

Proof. (a) In the case of S, this follows from Proposition 3.21 because $\mathbf{C}\mathscr{S}$ is a conical wall structure on the conical polyhedral pseudomanifold $\mathbf{C}B$. For R_∞, we apply Theorem 3.19.

(b) In Construction 4.11 we saw that $\mathfrak{Y}^\circ$ is a partial compactification of a $\mathbb{G}_m$-torsor over $\mathfrak{X}^\circ$ inside its corresponding line bundle. The weight with respect to the induced $\mathbb{G}_m$-action on $S = \Gamma(\mathfrak{Y}^\circ, \mathcal{O}_{\mathfrak{Y}^\circ})$ defines the $\mathbb{Z}$-grading on S. Of course, an element of S is homogeneous of degree d if and only if its local representatives in the rings $R_{\mathbf{C}\mathfrak{b}}$ and $R_{\mathbf{C}\mathfrak{u}}$ are homogeneous of degree d as defined in Construction 4.11.

The degree zero part of S has an $A[Q]/I$-basis ϑ_m with m an asymptotic monomial on $\mathbf{C}B$ of degree zero, which is hence an asymptotic monomial on B (Proposition 4.10). Embedding B as $B \times \{1\}$ into $\mathbf{C}B$ this shows that we can identify the degree zero part of S with the ring of global functions on $\mathfrak{X}^\circ$. In fact, the latter has an $A[Q]/I$-basis of canonical global functions with the same index set, and the multiplication rule only depends on broken lines for monomials of degree zero. These broken lines are parallel to $B \times \{1\}$, hence are in bijection with broken lines on B.

(c) Denote by $\mathcal{L}^\circ$ the invertible sheaf on $\mathfrak{X}^\circ$ associated to the dual of the $\mathbb{G}_m$-torsor $\mathfrak{L}^{\circ,\times} \to \mathfrak{X}^\circ$. By Construction 4.11 the linear space associated to $\mathcal{L}^\circ$ extends $\mathfrak{L}^\circ$ over the codimension one strata of $\mathfrak{X}^\circ$. Thus the sheaf of sections of the dual of $\mathfrak{L}^\circ$ agrees with the restriction of $\mathcal{L}^\circ$ to the complement of the codimension one strata of $\mathfrak{X}^\circ$. Denote by $\mathfrak{L} \to \mathfrak{X} = \mathrm{Proj}(S)$ the line bundle with sheaf of sections $\mathcal{O}_{\mathfrak{X}}(-1)$. Defining a morphism

$$\Phi : \mathfrak{X}^\circ \longrightarrow \mathfrak{X}$$

together with an isomorphism $\mathcal{L}^\circ \simeq \Phi^*\big(\mathcal{O}_{\mathfrak{X}}(1)\big)$ amounts to writing down a homomorphism

$$\phi : S \longrightarrow \bigoplus_{d\in\mathbb{N}} \Gamma\big(\mathfrak{X}^\circ, (\mathcal{L}^\circ)^{\otimes d}\big)$$

with the image of the first graded piece $S_1 \subseteq S$ generating $\mathcal{L}^\circ$ ([Gt1], §3.7). For the definition of ϕ note that S has an $A[Q]/I$-module basis of theta functions ϑ_m labelled by asymptotic monomials on $\mathbf{C}B$. Now the asymptotic monomials on $\mathbf{C}B$ and on $\overline{\mathbf{C}}B$ agree, and hence for any such m there is also a theta function ϑ'_m on $\mathfrak{L}^\circ$. By the definition of the local trivializations of $\mathfrak{L}^\circ$ (Construction 4.11), ϑ'_m with $\deg m = d$ is homogeneous of degree d in the fibre coordinate, and hence it defines a section of $(\mathcal{L}^\circ)^{\otimes d}$. Define ϕ by mapping ϑ_m to ϑ'_m. Note that ϕ is compatible with the multiplicative structures by comparison on $\mathfrak{L}^{\circ,\times} \subseteq \mathfrak{Y}^\circ$. To see that the image of S_1 generates $\mathcal{L}^\circ$, choose an interior codimension one cell $\rho \in \mathscr{P}$, $\mathfrak{b}$ a slab in ρ, let $v \in \rho$ be an integral point and m the associated asymptotic monomial on $\mathbf{C}B$ of degree 1 (Proposition 4.10). Then in the isomorphism $R_{\overline{\mathbf{C}}\mathfrak{b}} \simeq R_{\mathfrak{b}}[u, u^{-1}]$ induced by the choice of v (see Construction 4.11),

$$\vartheta_m = u + \cdots$$

with the dots standing for elements obtained by wall crossing. In particular, $\vartheta_m \equiv u$ modulo I_0, and hence ϑ_m generates $\mathcal{L}^\circ$ on the whole chart. A similar argument applies for the charts with ring $R^{\partial}_{\overline{\mathbf{C}}\mathfrak{u}}$.

It remains to show that $\Phi : \mathfrak{X}^\circ \to \mathfrak{X}$ is an open embedding. The following argument is analogous to the affine case of Propositions 3.17 and 3.21. By Proposition 2.19 the statement is true modulo I_0. There are two flat deformations of X_0, one given by $i_*\mathcal{O}_{\mathfrak{X}^\circ}$, the other by $\mathfrak{X} = \operatorname{Proj} S$. In both cases flatness follows by the criterion of [GHK1], Lemma 2.29. In fact, if $v \in \mathscr{P}$ is a vertex and $x \in X_0$ the corresponding zero-dimensional toric stratum, let $U_v \subseteq X_0$ be the affine open subset defined as the complement of toric strata disjoint from x. Denote by m_0 the asymptotic monomial of degree one defined by v. Then on U_v there is an $A[Q]/I_0$-module basis of regular functions of the form $\vartheta_m/\vartheta_{m_0}^d$, $d = \deg m$. Any of these lift to both deformations, as a quotient of theta functions. This proves flatness of both deformations. Moreover, by [GHK1], Lemma 2.30, the stated liftings are $A[Q]/I$-bases of the rings of regular functions. Since $\Phi|_{U_v}$ maps these liftings onto each other, we also obtain an isomorphism $(X_0, i_*\mathcal{O}_{\mathfrak{X}^\circ}) \simeq \mathfrak{X}$ by Lemma 3.18. In particular, $\mathfrak{X}^\circ \to \mathfrak{X}$ is an open embedding and $\mathfrak{X}$ has the stated uniqueness property.

(d) By (c) we can now identify $\mathfrak{X}^\circ$ with the complement of the codimension two strata in $\mathfrak{X}$. With this identification we have seen that $\mathfrak{L}^\circ$ is the restriction of the total space $\mathfrak{L}$ of $\mathcal{O}_{\mathfrak{X}}(-1)$ to $\mathfrak{X}^\circ$, with the codimension one strata of the zero section removed. Since, for a vertex v with corresponding asymptotic monomial m, ϑ_m yields a trivialization of $\mathcal{L}^\circ$ on $\mathfrak{X}^\circ \cap U_v$, we also see that $i_*\mathcal{L}^\circ = \mathcal{O}_{\mathfrak{X}}(1)$. For the statement on global functions on $\mathfrak{L}$ note the following sequence of inclusions and identifications

$$\Gamma(\mathfrak{L},\mathcal{O}_{\mathfrak{L}}) \subseteq \Gamma(\mathfrak{L}^\circ,\mathcal{O}_{\mathfrak{L}^\circ}) \subseteq \Gamma(\mathfrak{Y}^\circ,\mathcal{O}_{\mathfrak{Y}^\circ}) = \Gamma(\mathfrak{Y},\mathcal{O}_{\mathfrak{Y}}) = S. \tag{4.7}$$

Conversely, the $A[Q]/I$-module basis elements ϑ_m of S lift to global sections of $\mathcal{O}_{\mathfrak{X}}(d)$, $d=\deg m$, hence to an element of $\Gamma(\mathfrak{L},\mathcal{O}_{\mathfrak{L}})$, of fibrewise degree d. Hence all inclusion in (4.7) are indeed equalities.

The remaining statements follow from the usual correspondence between the projective variety associated to a $\mathbb{Z}$-graded ring generated in degree 1 and its affine cone. □

Remark 4.13. Following up on Remark 2.18 we now also obtain a partial completion $\mathfrak{D}\subseteq\mathfrak{X}$ of the divisor $\mathfrak{D}^\circ\subseteq\mathfrak{X}^\circ$. Indeed, each $\rho\subseteq\partial B$ defines a restriction map

$$\Gamma(\mathfrak{X}^\circ,\mathcal{O}_{\mathfrak{X}}(d)) \longrightarrow \Gamma\big(\mathfrak{D}^\circ_\rho,\mathcal{O}_{\mathfrak{X}}(d)|_{\mathfrak{D}^\circ_\rho}\big).$$

Taking the direct sum over d of the kernel of these maps defines a graded ideal $K_\rho\subseteq S$. It is then easy to see that K_ρ is a free $A[Q]/I$-module with generators defined by the theta functions ϑ_m with m an asymptotic monomial of $\mathbf{C}B$ but not of $\mathbf{C}\rho$. In particular, the quotient $S_\rho:=S/K_\rho$ is a free $A[Q]/I$-module with basis (the restrictions to D°_ρ of) the theta functions ϑ_m with m running over the asymptotic monomials of $\mathbf{C}\rho$. Moreover, this construction of S_ρ is obviously compatible with the construction of the homogeneous coordinate ring of $\mathfrak{D}^\circ_\rho$ via the wall structure $\mathscr{S}_\rho$ on ρ in Remark 2.18. In particular, $\mathfrak{D}_\rho:=\operatorname{Proj}(S_\rho)$ defines a partial completion of $\mathfrak{D}^\circ_\rho$. Note however that by our definition of $\Delta\subseteq B$ the complement of $\mathfrak{D}^\circ_\rho$ in $\mathfrak{D}_\rho$ is a union of divisors rather than of codimension two subsets as for $\mathfrak{X}^\circ\subseteq\mathfrak{X}$.

The divisor $\mathfrak{D}\subseteq\mathfrak{X}$ is then defined as the scheme theoretic union of the $\mathfrak{D}_\rho$, that is, it is the closed subscheme of $\mathfrak{X}$ given by the homogeneous ideal $\bigcap_\rho K_\rho\subseteq S$.

Remark 4.14. It is also easy to treat the completion $\mathbb{P}(\mathcal{O}_{\mathfrak{X}}(1)\oplus\mathcal{O}_{\mathfrak{X}})=\mathbb{P}(\mathcal{O}_{\mathfrak{X}}\oplus\mathcal{L})$ of $\mathfrak{L}$ to a $\mathbb{P}^1$-bundle in the current framework. For an integer $a>1$ consider the polyhedral pseudomanifold

$$\mathbf{C}_{[1,a]}B := \big\{(x,h)\in\mathbf{C}B \,\big|\, h\in[1,a]\big\},$$

and write $\mathfrak{P}^\circ$ for the associated flat scheme over $A[Q]/I$. Then for any chamber $\mathfrak{u}$ for $\mathscr{S}$ and an integral point $v\in\mathfrak{u}$ there are two non-interior slabs $\mathfrak{u}\times\{1\}$ and $\mathfrak{u}\times\{a\}$ for the induced wall structure on $\mathbf{C}_{[1,a]}B$. These give rise to two charts for $\mathfrak{P}^\circ$,

$$R_{\mathfrak{u}\times\{1\}} \xrightarrow{\simeq} R_{\mathfrak{u}}[u],\quad R_{\mathfrak{u}\times\{a\}} \xrightarrow{\simeq} R_{\mathfrak{u}}[v].$$

Clearly, $\mathfrak{P}^\circ$ contains the $\mathbb{G}_m$-torsor $\mathfrak{L}^{\circ,\times}$ as an open dense subscheme and $u|_{\mathfrak{Y}^\circ}=(v|_{\mathfrak{Y}^\circ})^{-1}$ generate this $\mathbb{G}_m$-torsor locally. Thus $\mathfrak{P}^\circ$ is an open subscheme of the

$\mathbb{P}^1$-bundle $\mathbb{P}(\mathcal{O}_{\mathfrak{X}^\circ} \oplus \mathcal{L}^\circ)$ with complement two copies of the codimension one strata of $\mathfrak{X}^\circ$. It extends to the flat deformation $\mathbb{P}(\mathcal{O}_{\mathfrak{X}} \oplus \mathcal{O}_{\mathfrak{X}}(1))$ of $\mathbb{P}(\mathcal{O}_{X_0} \oplus \mathcal{O}_{X_0}(1))$ by pushing forward the sheaf of regular functions.

Note that the integral length $a-1$ of the interval $[1,a]$ agrees with the degree (as a line bundle over $\mathbb{P}^1$) of $\mathcal{O}_{\mathbb{P}(\mathcal{O}_{\mathfrak{X}} \oplus \mathcal{O}_{\mathfrak{X}}(1))}(1)$ on a fibre of $\mathbb{P}(\mathcal{O}_{\mathfrak{X}} \oplus \mathcal{O}_{\mathfrak{X}}(1))$. The lower and upper boundaries of $\mathbf{C}_{[1,a]}B$ represent the two distinguished sections of $\mathbb{P}(\mathcal{O}_{\mathfrak{X}} \oplus \mathcal{O}_{\mathfrak{X}}(1))$, with the lower boundary giving the contractible ("negative") section. Interestingly, the restriction of the polarization to the other section does not produce the polarization on $\mathfrak{X}$ but its a-fold multiple.

4.4. **The action of the relative torus.** Another feature of the construction in many cases is the existence of a canonical action of a large algebraic torus on $\mathfrak{X}$ as a projective scheme. Our theta functions generate isotypical components for the induced action on $\Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}(d))$.

We begin by identifying the group of automorphisms of X_0 over $A[Q]/I_0$. This result is of motivational character for explaining the role of $\mathrm{PL}(B)^*$ both in the present subsection as well as in §A.3. Here we write $\mathrm{PL}(B) = \mathrm{PL}(B,\mathbb{Z})$ and $\mathrm{PL}(B)^* = \mathrm{Hom}(\mathrm{PL}(B),\mathbb{Z})$ for brevity. Note that $\mathrm{PL}(B)$ depends only on the affine structure on the interiors of the maximal cells, just as the central fibre X_0.

Proposition 4.15. *The connected component of the identity of* $\mathrm{Aut}_{A[Q]/I_0}(X_0)$ *is the torus over* $\mathrm{Spec}(A[Q]/I_0)$ *with character lattice* $\mathrm{PL}(B)^*$.

Proof. Write $S = A[Q]/I_0$ for brevity. Since the irreducible components of X_0 are toric varieties over S labelled by $\mathscr{P}_{\max}$, the connected component of the identity of $\mathrm{Aut}_S(X_0)$ is a closed subgroup of a product of tori $\mathbb{G}_m^n$, one for each maximal cell of $\mathscr{P}$. Intrinsically, the torus for the component labelled by $\sigma \in \mathscr{P}_{\max}$ is $\mathbf{T}_\sigma := \mathrm{Spec}\big(S[\Lambda_\sigma]\big)$. For any facet $\rho \subseteq \sigma$ the inclusion $\Lambda_\rho \subseteq \Lambda_\sigma$ defines an epimorphism $\mathbf{T}_\sigma \to \mathbf{T}_\rho$, with $\mathbf{T}_\rho = \mathrm{Spec}\big(S[\Lambda_\rho]\big)$ the torus for the codimension one stratum labelled by ρ. Compatibility of the actions for $\sigma, \sigma' \in \mathscr{P}_{\max}$ adjacent to $\rho \in \mathscr{P}^{[n-1]}$ restricts the automorphism group of the union of the corresponding components to the fibre product

$$\mathbf{T}_\sigma \times_{\mathbf{T}_\rho} \mathbf{T}_{\sigma'} = \mathrm{Spec}\big(S[\Lambda_\sigma \oplus_{\Lambda_\rho} \Lambda_{\sigma'}]\big).$$

By trivially writing $\Lambda_\sigma = \mathrm{PL}(\sigma,\mathbb{Z})^*$, the fibred sum can conveniently be interpreted as the dual of the group of $\mathbb{Z}$-valued piecewise linear functions on $\sigma \cup \sigma'$.

Globally we need to take the limit of the category with morphisms $T_\sigma \to T_\rho$ for $\rho \in \mathscr{P}^{[n-1]}$, $\sigma \in \mathscr{P}_{\max}$, $\rho \subseteq \sigma$. In fact, since X_0 is S_2 it suffices to check compatibility of the actions on the irreducible components in codimension one. Dually this leads to the colimit of $\Lambda_\rho \to \Lambda_\sigma$, which is $\mathrm{PL}(B)^*$. □

Let us now discuss the general procedure for obtaining a torus action. For the character lattice of the acting torus take a finitely generated free abelian group Γ. For the various Γ-graded rings, $\deg_\Gamma$ always denotes the degree, as a map from the set of homogeneous elements to Γ. Assume we have a Γ-grading on our ground ring $A[Q]$ induced by gradings on A and Q and that $I \subseteq A[Q]$ is a homogeneous ideal. Denote by $\delta_Q : Q \to \Gamma$ the homomorphism defining the grading on the monomials of $A[Q]$ and by $A_0^Q \subseteq A[Q]$ the degree zero subring. Our torus is

$$\mathbb{T} := \operatorname{Spec}\big(A_0^Q[\Gamma]\big).$$

Then $\mathbb{T}$ acts naturally on $\operatorname{Spec} A[Q]$, defined on the ring level by

$$A[Q] \longrightarrow A_0^Q[\Gamma] \otimes_{A_0^Q} A[Q], \quad a \longmapsto z^{\deg_\Gamma(a)} \otimes a,$$

the map written on a homogeneous element a. The fixed locus of the action is $\operatorname{Spec} A_0^Q$.

For lifting this action to $\mathfrak{X}$ recall from Proposition 1.12 the universal monoid $Q_0 = \mathrm{MPA}(B, \mathbb{N})^\vee$ and the homomorphism $h : Q_0 \to Q$ defining the given Q-valued MPA-function φ. By the explicit description in Proposition 1.9, one has $Q_0^{\mathrm{gp}} = \mathrm{MPA}(B, \mathbb{Z})^*$. Hence the dual of the map $\mathrm{PL}(B) \to \mathrm{MPA}(B, \mathbb{Z})$ defines a homomorphism

$$g : Q_0 \hookrightarrow Q_0^{\mathrm{gp}} = \mathrm{MPA}(B, \mathbb{Z})^* \longrightarrow \mathrm{PL}(B)^*.$$

We now assume given a further homomorphism $\delta_B : \mathrm{PL}(B)^* \to \Gamma$ fitting into the following commutative diagram.

$$\begin{array}{ccc} Q_0 & \xrightarrow{\ g\ } & \mathrm{PL}(B)^* \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle \delta_B} \\ Q & \xrightarrow{\ \delta_Q\ } & \Gamma. \end{array} \tag{4.8}$$

This data provides a grading of our monomials as follows. Recall that a monomial m is an integral tangent vector on $\mathbb{B}_\varphi$ at a point x of $\varphi(B_0) \subseteq \mathbb{B}_\varphi$ and that $\pi : \mathbb{B}_\varphi \to B$ denoted the projection. Assuming that $\overline{m} = \pi_*(m)$ points from x into the tangent wedge of a cell τ at x, the directional derivative in the direction of $\overline{m}$ defines an element $\nabla_{\overline{m}} \in \mathrm{PL}(B)^*$. Said differently, for $\psi \in \mathrm{PL}(B)$ the restriction to τ defines an element of $\check{\Lambda}_\tau$, and we define

$$\nabla_{\overline{m}}(\psi) := (\psi|_\tau)(\overline{m}). \tag{4.9}$$

A monomial m also yields an element $m_Q \in Q$, by subtracting the lift of $\overline{m}$ to $\mathbb{B}_\varphi$ via the piecewise affine section $\varphi : B \to \mathbb{B}_\varphi$ of π. Note that in the canonical identification (1.10) in the interior of a maximal cell we have $m = (\overline{m}, m_Q)$. The Γ-*degree* of m or of z^m is now defined as

$$\deg_\Gamma(m) := \delta_Q(m_Q) + \delta_B(\nabla_{\overline{m}}).$$

We have thus made the rings R_σ from (2.5) into Γ-graded rings. The basic result of this subsection is that the whole construction is Γ-graded provided all functions $f_{\mathfrak{p}}$ given by the walls are homogeneous of degree zero.

Definition 4.16. Assume $A[Q]/I$ and the monomials on B_0 are graded by a finitely generated free abelian group Γ via a homomorphism $\delta_B : \mathrm{PL}(B)^* \to \Gamma$ making (4.8) commutative, as just described. Let $\mathscr{S}$ be a wall structure on $(B, \mathscr{P})$. We say that $\mathscr{S}$ is a *homogeneous wall structure* if all functions $f_{\mathfrak{p}}$ defining walls are homogeneous of degree 0.

Theorem 4.17. *Let $\mathscr{S}$ be a consistent homogeneous wall structure on $(B, \mathscr{P})$. Then the algebraic torus $\mathbb{T} = \mathrm{Spec}\left(A_0^Q[\Gamma]\right)$ acts equivariantly on the flat family $\mathfrak{X} \to \mathrm{Spec}(A[Q]/I)$ from Theorem 4.12.*

Proof. As all fibres of $\mathfrak{X} \to \mathrm{Spec}(A[Q]/I)$ satisfy Serre's condition S_2 by Proposition 2.7, Lemma 3.18 implies it is enough to prove the statement after restricting to the complement $\mathfrak{X}^\circ \subseteq \mathfrak{X}$ of codimension two strata. Recall that $\mathfrak{X}^\circ$ is covered by rings of the form $\mathrm{Spec}\, R_{\mathfrak{u}}$, $\mathrm{Spec}\, R_{\mathfrak{u}}^{\partial}$ and $\mathrm{Spec}\, R_{\mathfrak{b}}$ for chambers $\mathfrak{u}$ and slabs $\mathfrak{b}$ for $\mathscr{S}$, with the gluing coming from canonical embeddings and automorphisms governed by wall crossing. For a chamber $\mathfrak{u}$ contained in a maximal σ we have already seen that $R_{\mathfrak{u}} = R_\sigma$ is naturally Γ-graded by the grading of monomials.

For the Γ-grading of the rings $R_{\mathfrak{b}}$ (2.17) we have to check that $Z_+Z_- - f_{\mathfrak{b}} z^{\kappa_{\underline{\rho}}}$ is homogeneous. Again, since $f_{\mathfrak{b}}$ is homogeneous of degree zero this statement is equivalent to

$$\deg_\Gamma(Z_+) + \deg_\Gamma(Z_-) = \deg_\Gamma\left(z^{\kappa_{\underline{\rho}}}\right).$$

To prove this equality recall that if m_+, m_- are the tangent vectors on $\mathbb{B}_\varphi$ with $Z_+ = z^{m_+}$, $Z_- = z^{m_-}$, then $\deg_\Gamma(m_\pm) = \delta_B(\nabla_{\overline{m}_\pm})$, while $\deg_\Gamma(z^{\kappa_{\underline{\rho}}}) = \delta_Q(\kappa_{\underline{\rho}})$. But $\nabla_{\overline{m}_+} + \nabla_{\overline{m}_-}$ is the linear functional on $\mathrm{PL}(B)$ given by $\psi \mapsto \kappa_{\underline{\rho}}(\psi)$. This is precisely the image of $e_{\underline{\rho}} \in Q_0$ under the map $Q_0 \to \mathrm{PL}(B)^*$ (see Proposition 1.12). But $\kappa_{\underline{\rho}} \in Q$ is also the image of $e_{\underline{\rho}}$. Thus the claimed equality now follows from the commutativity of (4.8).

The same degree computation shows homogeneity of the localization maps $R_{\mathfrak{b}} \to R_{\mathfrak{u}}$ of (2.18) for a slab $\mathfrak{b}$ and an adjacent chamber $\mathfrak{u}$. If $\mathfrak{u}$ is a boundary chamber the localization $R_{\mathfrak{u}}^{\partial} \to R_{\mathfrak{u}}$ of (2.18) is localization at a homogeneous monomial, so respects the grading. The automorphisms $\theta_{\mathfrak{p}} : R_{\mathfrak{u}} \to R_{\mathfrak{u}'}$ of (2.19) associated to crossing a wall $\mathfrak{p}$ separating chambers $\mathfrak{u}, \mathfrak{u}'$ manifestly respects the grading since by hypothesis $f_{\mathfrak{p}}$ is homogeneous of degree zero. Finally, for slabs $\mathfrak{b}, \mathfrak{b}'$ separated by a joint, the isomorphism $R_{\mathfrak{b}} \to R_{\mathfrak{b}'}$ from (2.22) is induced by a composition of wall crossing homomorphisms and localizations, hence also respects the grading. □

Remark 4.18. For lifting the statement of Theorem 4.17 to the total space $\mathfrak{L}\to\mathfrak{X}$ of the line bundle in Theorem 4.12, note that there is a bijection between the group $\mathrm{PA}(B,\mathbb{Z})$ of piecewise affine functions on B and $\mathrm{PL}(\mathbf{C}B,\mathbb{Z})$. Moreover, since there is a bijection between the interior codimension one cells of B and the interior codimension one cells of $\mathbf{C}B$, we can identify the universal monoids Q_0 of B and of $\mathbf{C}B$. Thus taking for δ_B a homomorphism $\mathrm{PA}(B,\mathbb{Z})^*\to\Gamma$ both in (4.8) and in the statement of Theorem 4.17, the action of $\mathbb{T}$ lifts to $\mathfrak{L}$. Note that the condition on homogeneity of the wall functions $f_{\mathfrak{p}}$ can nevertheless be checked on B by the definition of lifted wall structures (Definition 4.7).

Remark 4.19. In §5.2 the construction of $\mathfrak{X}$ will be modified by the introduction of gluing data. They are given by homomorphisms $s_{\sigma\underline{\rho}}:\Lambda_\sigma\to A^*$. For projective open gluing data (Definition 5.13) the analogue of Theorem 4.12 holds. In this modified setup Theorem 4.17 holds true provided the $s_{\sigma\underline{\rho}}$ take values in degree zero, for then the localization homomorphisms $\chi_{\mathfrak{b},\mathfrak{u}}$ remain homogeneous. The rest of the construction is untouched.

Without a projectivity assumption one only obtains $\mathfrak{X}^\circ\to\mathrm{Spec}(A[Q]/I)$ and, again assuming the gluing data to be homogeneous of degree zero, a torus action on $\mathfrak{X}^\circ$.

Example 4.20. In [GHKS] (see Examples 1.3,2 and 1.11,2) one takes $A=\Bbbk$ and for Q (denoted P in loc.cit.) a toric submonoid of $N_1(\mathcal{Y}/T)=Q^{\mathrm{gp}}$ containing $\mathrm{NE}(\mathcal{Y}/T)$, the group of effective 1-cycles of the mirror family $\mathcal{Y}\to T$ of K3 surfaces. The character lattice Γ of $\mathbb{T}$ is $\mathbb{Z}^{B(\mathbb{Z})}$, the free abelian group generated by the integral points of B. In this example every integral point v of B is a vertex, hence labels an irreducible component $Y_v\subseteq\mathcal{Y}_0$. The Γ-grading on $\Bbbk[Q]$ is defined by intersection theory on $\mathcal{Y}$:

$$\delta_Q: N_1(\mathcal{Y}/T)\longrightarrow\Gamma,\quad C\longmapsto\sum_{v\in B(\mathbb{Z})}(C\cdot Y_v)\cdot e_v,$$

$e_v\in\Gamma$ the canonical basis vector labelled by v.

As for $\delta_{\mathbf{C}B}:\mathrm{PA}(B,\mathbb{Z})^*\to\Gamma$ (Remark 4.18) note that in this case all maximal cells are standard simplices. Hence $\mathrm{PA}(B,\mathbb{Z})=\mathbb{Z}^{B(\mathbb{Z})}$. We take $\delta_B=\mathrm{id}_{\mathbb{Z}^{B(\mathbb{Z})}}$.

To check commutativity of (4.8) it suffices to trace the generators of Q_0,

$$\kappa_\rho:\mathrm{MPA}(B,\mathbb{N})\longrightarrow\mathbb{N},\quad\psi\longmapsto\kappa_\rho(\psi),$$

measuring the kink of an MPA function along an edge $\rho\in\mathscr{P}$, through the diagram. The image of κ_ρ in $\mathrm{PA}(B,\mathbb{Z})^*=\mathrm{PL}(\mathbf{C}B,\mathbb{Z})^*$ measures the kink of a piecewise affine function ψ along ρ, still denoted $\kappa_\rho(\psi)$. Let v_0,v_1 be the vertices of ρ and let v_2,v_3 be the remaining vertices of the two triangles containing ρ.

Denote by D^2_{vw} the self-intersection number of the double curve $Y_v \cap Y_w$ inside Y_w. A straightforward computation in the affine chart (1.2) shows

$$\kappa_\rho(\psi) = \psi(v_2) + \psi(v_3) - (D^2_{v_1v_0} + 2)\psi(v_0) + D^2_{v_1v_0}\psi(v_1).$$

Noting that $D^2_{v_0v_1} + D^2_{v_1v_0} = -2$ we see that $\kappa_\rho \in Q_0$ maps to the symmetric expression

$$e_{v_2} + e_{v_3} - (D^2_{v_1v_0} + 2)e_{v_0} - (D^2_{v_0v_1} + 2)e_{v_1} \in \mathrm{PA}(B, \mathbb{Z})^*. \tag{4.10}$$

On the other hand, going via $Q \subseteq A_1(\mathcal{Y}/T)$ maps κ_ρ first to $C = Y_{v_0} \cap Y_{v_1} \subseteq \mathcal{Y}$ and then on to $\sum_v (C \cdot Y_v) \cdot e_v$. Now C intersects Y_{v_2} and Y_{v_3} transversely, while

$$C \cdot Y_{v_0} = \deg_C \mathcal{O}_{\mathcal{Y}}(Y_{v_0}) = -\deg_C \mathcal{O}_{\mathcal{Y}}(Y_{v_1} + Y_{v_2} + Y_{v_3}) = -2 - \deg_C \mathcal{O}_{Y_{v_0}}(C) = -2 - D^2_{v_1v_0},$$

and similarly for $C \cdot Y_{v_1}$. Here we used that $\mathcal{O}_{\mathcal{Y}}(\sum_v Y_v) = \mathcal{O}_{\mathcal{Y}}$. Since C is disjoint from all other Y_v we obtain the same expression as in (4.10).

Homogeneity of the wall functions for the walls emanating from vertices follows by an a priori argument. The remaining walls will be seen to be homogeneous because the scattering procedure via the Kontsevich-Soibelman lemma manifestly respects the grading.

The somewhat complementary case of GS-type singularities will be treated in §A.3.

4.5. **Jagged paths.** An alternative point of view on the construction of our theta functions from §4.3 works directly on B rather than on $\mathbf{C}B$. Recall that $\mathbf{C}B = (B \times \mathbb{R}_{\geq 0})/(B \times \{0\})$ topologically. The projection to the second factor induces an affine map

$$h : \mathbf{C}B \longrightarrow \mathbb{R}_{\geq 0},$$

the height functions, while the projection to the second factor (the radial directions) defines a non-affine retraction

$$\kappa : \mathbf{C}B \setminus \{O\} = h^{-1}(\mathbb{R}_{\geq 0}) \longrightarrow B.$$

The projection of broken lines via κ leads to the notion of *jagged paths*. The image of a broken line still consists of a union of straight line segments in B, but the slopes need not be rational since the projection κ is not affine linear. The notion of jagged paths predates the notion of broken lines. It had been discussed early in 2007 in a project on tropical Morse theory of the first and fourth authors of this paper jointly with Mohammed Abouzaid. Tropical Morse theory is a tropical version of Floer theory for Lagrangian sections of the SYZ fibration, see [DBr], §8.4 and [GrSi6].

We begin by reexamining the affine geometry of $\mathbf{C}B$ from §4.1 and §4.2. Denote by

$$j : B \longrightarrow \mathbf{C}B$$

the identification of B with $B \times \{1\} \subseteq \mathbf{C}B$.

First we want to interpret the tangent vectors on $\mathbf{C}B_0$ purely in terms of the affine geometry of B_0.

Lemma 4.21. *There is a canonical isomorphism $\mathcal{A}\!f\!f(B_0, \mathbb{Z})^* \simeq j^*(\Lambda_{\mathbf{C}B_0})$.*

Proof. It suffices to establish this isomorphism for an n-dimensional lattice polyhedron $\sigma \subseteq \mathbb{R}^n$. For $x \in \sigma$ an integral point there is a canonical identification

$$\mathcal{A}\!f\!f(\sigma, \mathbb{Z})_x = \mathrm{Aff}(\Lambda_x, \mathbb{Z}) \xrightarrow{\simeq} \check{\Lambda}_x \oplus \mathbb{Z},$$

mapping $0 \oplus \mathbb{Z}$ to the constant functions and $\check{\Lambda}_x \oplus 0$ to the affine functions vanishing at x. Dualizing gives

$$\mathcal{A}\!f\!f(\sigma, \mathbb{Z})^*_x = \Lambda_x \oplus \mathbb{Z}. \tag{4.11}$$

The latter is canonically isomorphic to $\Lambda_{\mathbf{C}\sigma,(x,1)}$ by mapping $(0, 1) \in \Lambda_x \oplus \mathbb{Z}$ to ∂_r, the tangent vector in the radial direction, which is integral at the integral point x, while Λ_x is canonically embedded into $\Lambda_{\mathbf{C}\sigma,(x,1)}$ via j_*.

Changing coordinates clearly respects this identification of Λ_x with those homomorphisms $\mathcal{A}\!f\!f(\sigma, \mathbb{Z})_x \to \mathbb{Z}$ that vanish on constant functions, that is, which factor over $\check{\Lambda}_x$. To generate $\mathcal{A}\!f\!f(\sigma, \mathbb{Z})^*_x$ it suffices to take in addition the evaluation homomorphism $\mathrm{ev}_p : \mathcal{A}\!f\!f(\sigma, \mathbb{Z})_x \to \mathbb{Z}$ at x. Under the isomorphism (4.11) this element of $\mathcal{A}\!f\!f(\sigma, \mathbb{Z})^*_x$ corresponds to $(0, 1)$, hence it maps to the primitive radial tangent vector $\partial_r \in \Lambda_{\mathbf{C}\sigma,(x,1)}$. Parallel transport to a nearby integral point $y = x + v \in \sigma$, $v \in \Lambda_x$, takes $(\alpha, c) \in \check{\Lambda}_y \oplus \mathbb{Z} = \mathrm{Aff}(\Lambda_y, \mathbb{Z})$ to $(\alpha, c - \langle \alpha, v\rangle) \in \mathrm{Aff}(\Lambda_x, \mathbb{Z})$. Thus $\mathrm{ev}_x = (-v, 1) \in \Lambda_y \oplus \mathbb{Z}$. This result agrees with the parallel transport of ∂_r at $(x, 1)$ to $(y, 1)$ in $\mathbf{C}\sigma$ (Proposition 4.2). ☐

Lemma 4.21 demonstrates that the tangent sequence for B_0 in $\mathbf{C}B_0$

$$0 \longrightarrow \Lambda \longrightarrow j^*\Lambda_{\mathbf{C}B_0} \xrightarrow{h_*} \underline{\mathbb{Z}} \longrightarrow 0 \tag{4.12}$$

agrees with the dual of (1.1),

$$0 \longrightarrow \Lambda \longrightarrow \mathcal{A}\!f\!f(B_0, \mathbb{Z})^* \xrightarrow{\deg} \underline{\mathbb{Z}} \longrightarrow 0. \tag{4.13}$$

The homomorphism deg can be characterized by the property that $\deg(\tilde{m})$ for $\tilde{m} \in \mathcal{A}\!f\!f(\sigma, \mathbb{Z})^*_x$ is the integer d with $\tilde{m} - d \cdot \mathrm{ev}_x \in \Lambda_x$, again for $\sigma \in \mathscr{P}$ a maximal cell. Said differently, $\deg(\tilde{m}) = \tilde{m}(1)$, the value of $\tilde{m}$ at the constant affine function 1. Note that the sequence also shows that $\mathcal{A}\!f\!f_d(B_0, \mathbb{Z}) := \deg^{-1}(d)$ is a Λ-torsor.

Given a Q-valued MPA-function φ the correspondence can be applied to the cone over $\mathbb{B}_\varphi$ (Construction 1.14). Note that $\mathbf{C}\mathbb{B}_\varphi = \mathbb{B}_{\mathbf{C}\varphi}$. In turn, we have a definition of monomials at a point $(x,h) \in \mathbf{C}B_0$ as elements in $\mathcal{A}\!f\!f(\mathbb{B}_\varphi,\mathbb{Z})^*_x$. Here we use parallel translation in the radial direction to reduce to the case $h=1$ treated in Lemma 4.21 and the discussion following it. In other words, $\mathcal{A}\!f\!f(\mathbb{B}_\varphi,\mathbb{Z})^* = j^*\mathcal{P}_{\mathbf{C}B_0}$ for $\mathcal{P}_{\mathbf{C}B_0}$ the sheaf on $\mathbf{C}B_0$ according to Definition 1.15. Denote the pullback via the secton $\varphi : B_0 \to \mathbb{B}_\varphi$ of either of these sheaves by $\tilde{\mathcal{P}}$, and the corresponding subsheaf of monomials by $\tilde{\mathcal{P}}^+$ (Definition 2.6). The homomorphism deg agrees with the grading of the monomials on $\mathbf{C}B_0$ defined before Proposition 4.10. The following diagram encapsulates the above discussion:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \underline{Q}^{\mathrm{gp}} & \xrightarrow{=} & \underline{Q}^{\mathrm{gp}} & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \mathcal{P} & \longrightarrow & \tilde{\mathcal{P}} & \xrightarrow{\deg} & \underline{\mathbb{Z}} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \pi_*} & & \downarrow{\scriptstyle \pi_*} & & \downarrow{\scriptstyle =} & & \\
0 & \longrightarrow & \Lambda & \longrightarrow & \mathcal{A}\!f\!f(B,\mathbb{Z})^* & \longrightarrow & \underline{\mathbb{Z}} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
$$

We now have a generalization of the notion of monomials on B to higher degree.

Definition 4.22. Denote by $\tilde{\mathcal{P}}_d := \deg^{-1}(d) \subseteq \tilde{\mathcal{P}}$. A *monomial of degree d* at $x \in B_0$ is a formal expression az^m with $a \in A$ and $m \in (\tilde{\mathcal{P}}_d)_x$. If m is a monomial at $x \in B_0$ we still denote by $\overline{m} \in \mathcal{A}\!f\!f(B_0,\mathbb{Z})^* = \Lambda_{\mathbf{C}B_0,(x,1)}$ the image induced by the affine projection $\mathbb{B}_{\mathbf{C}\varphi} \to \mathbf{C}B_0$.

Monomials of degree zero are the monomials from Definition 2.8 that we worked with so far. To construct a section of $\mathcal{L}^d$ one restricts to monomials of degree d. Note also that the notion of transport of monomials (Definition 3.2) readily generalizes to monomials of higher degree, simply by working on $\mathbf{C}B_0$ locally.

To translate the condition of constant velocity on the domains of affine linearity of a broken line (Definition 3.3,(1)) on $\mathbf{C}B_0$ to B_0 we need to compose the map $m \mapsto \overline{m}$ with the differential $\kappa_* : \mathcal{T}_{\mathbf{C}B_0,(x,1)} \to \mathcal{T}_{B_0,x}$. Here $\mathcal{T}_{B_0}$ is the sheaf of *differentiable* vector fields on B_0, and similarly on $\mathbf{C}B_0$. The resulting homomorphism is denoted

$$\mathbf{vect} : \tilde{\mathcal{P}} \longrightarrow \mathcal{T}_{B_0}, \quad m \longmapsto \kappa_*(\overline{m}).$$

The image of a local section of $\tilde{\mathcal{P}}$ under **vect** provides a flat section of $\mathcal{T}_{B_0}$ with respect to the *affine* connection on T_{B_0}, see the discussion before Proposition 4.2.

An alternative description of **vect** is by noting that it factors via the map $m \to \overline{m}$ with target in $\mathcal{A}\!f\!f(B_0,\mathbb{Z})^* \subseteq \mathcal{A}\!f\!f(B_0,\mathbb{R})^*$ and

$$\mathrm{vect} : \mathcal{A}\!f\!f(B_0,\mathbb{R})^* \longrightarrow \mathcal{T}_{B_0}.$$

This latter map sends a linear functional on $\mathcal{A}\!f\!f(B,\mathbb{R})_x$ for $x \in B_0$ to its restriction to the subspace of germs of affine linear functions vanishing at x. In fact, the dual of the space of germs of affine linear functions vanishing at x is the tangent space to B at x.

Definition 4.23. A (normalized) *jagged path* of *degree* d for a wall structure $\mathscr{S}$ on $(B,\mathscr{P})$ is a proper continuous map

$$\gamma : [0,t_r] \to B$$

with $\gamma\big((0,t_r)\big) \subseteq B_0$ and disjoint from any joints of $\mathscr{S}$, along with a sequence $0 = t_0 < t_1 < \ldots < t_r$ for some $r \geq 1$ with $\gamma(t_i) \in |\mathscr{S}|$ for $i = 1,\ldots,r-1$, and for $i = 1,\ldots,r$ monomials $a_i z^{m_i}$ of degree d defined at all points of $\gamma([t_{i-1},t_i])$, subject to the following conditions.

(1) $\gamma|_{(t_{i-1},t_i)}$ is a map with image disjoint from $|\mathscr{S}|$, hence contained in the interior of a unique chamber $\mathfrak{u}_i$ of $\mathscr{S}$, and $\gamma'(t) = -\mathbf{vect}(m_i)$ for all $t \in (t_{i-1},t_i)$.
(2) For each $i = 1,\ldots,r-1$ the monomial $a_{i+1}z^{m_{i+1}}$ is a result of transport of $a_i z^{m_i}$ from $\mathfrak{u}_i$ to $\mathfrak{u}_{i+1}$.
(3) $a_1 = 1$, $m_1 = d\cdot\varphi_*(\mathrm{ev}_{\gamma(0)})$, $\gamma(0) \in B\big(\frac{1}{d}\mathbb{Z}\big)$.

The *type* of γ is the tuple of all $\mathfrak{u}_i$ and m_i. As for broken lines we suppress the data t_i, a_i, m_i when talking about jagged paths, but introduce the notation

$$a_\gamma := a_r, \quad m_\gamma := m_r.$$

In (3) the push-forward φ_* is understood by first restricting φ to a maximal cell $\sigma \in \tilde{\mathscr{P}}$ containing $\gamma\big((0,t_1)\big)$ to obtain an affine map $\mathrm{Int}\,\sigma \to \mathbb{B}_\varphi$.

Comparing to the notion of broken line the one point to emphasize is that while a broken line has an asymptotic vector (Remark 3.4,1), a jagged path has an initial point $\gamma(0)$.

Proposition 4.24. *Let $\mathscr{S}$ be a wall structure on the polyhedral pseudomanifold $(B,\mathscr{P})$. Then the projection $\kappa : \mathbf{C}B \to B$ induces a bijection between the set of broken lines on $\mathbf{C}B$ for $\mathbf{C}\mathscr{S}$ with endpoint p and the set of jagged paths on B for $\mathscr{S}$ with endpoint $\kappa(p)$. If β is a broken line on $\mathbf{C}B$ with asymptotic monomial $\overline{m}$ of degree d, the initial point of the associated jagged path is the point $x \in B(\frac{1}{d}\mathbb{Z})$ corresponding to $\overline{m}$ according to Proposition 4.10.*

Proof. This follows directly from the definitions. □

Having related the notion of broken line on $\mathbf{C}B$ to the notion of jagged path on B it is now immediate to express all results in §4.3 in terms of jagged paths.

5. Additional parameters

So far X_0 is the pull-back of a scheme over $\operatorname{Spec}\mathbb{Z}$ to $\operatorname{Spec}\big(A[Q]/I_0\big)$. Moreover, by the definition of the rings $R_{\underline{\rho}}$ in (2.11) the closed subscheme of $\operatorname{Spec}\big(A[Q]/I\big)$ defined by I_0 describes a trivial deformation. This is enough for certain cases, for example to describe projective deformations of certain degenerate K3 surfaces with all irreducible components copies of $\mathbb{P}^2$ [GHKS], but in general it is important to include also non-trivial locally trivial[11] deformations. For example, in [GrSi3], §5.2 we describe a locally trivial family X_0 parametrized by the algebraic torus $\operatorname{Spec}\big(\Bbbk[\Gamma]\big)$ with Γ the quotient by the torsion subgroups of the abelian group $H^1(B, i_*\Lambda)$, $i : B_0 \hookrightarrow B$. This family comes with a log structure and is versal as a family of log schemes keeping the singularity structure, and it usually is non-trivial as a family of schemes. Assuming projectivity, [GrSi4] yields a deformation $\mathfrak{X}$ of X_0 much of the same form as the construction presented here, but involving parameters in the localization morphisms. To keep the presentation simple we chose not to include these in the discussion up to this point. The purpose of this section is finally to include these additional parameters.

5.1. **Twisting the construction.** We begin by a general consideration on including additional parameters abstractly. In this general framework $A[Q]$ also includes these parameters. The reader is advised to think of $\operatorname{Spec} A$ as the space of such gluing parameters, although this may not be strictly true in practice.

In the new setup the definition of the sheaf of rings $A[\mathcal{P}]$, the notion of wall structure and the rings $R_{\mathfrak{u}}$ and $R_{\mathfrak{b}}$ are as before. The only data that has to be changed in our construction is the localization morphism from the ring for a slab $\mathfrak{b}$ to an adjacent chamber $\mathfrak{u}$, which previously was defined canonically in terms of the affine geometry of B_0. For each such pair $(\mathfrak{b}, \mathfrak{u})$ we now have as additional datum a homomorphism of $A[Q]/I$-algebras

$$\chi_{\mathfrak{b},\mathfrak{u}} : R_{\mathfrak{b}} \longrightarrow R_{\mathfrak{u}}.$$

At this level of generality there are no restrictions on $\chi_{\mathfrak{b},\mathfrak{u}}$. This new definition of the transition between $R_{\mathfrak{b}}$ and $R_{\mathfrak{u}}$ changes also the notion of consistency in codimension one (Definition 2.14) and the definition of the isomorphism $\theta_{\mathfrak{j}}$ between rings $R_{\mathfrak{b}}$ associated to crossing a codimension one joint (2.22). Under the

[11] Recall that a deformation is called *locally trivial* if the total space has an étale covering by open subsets of trivial families.

assumption of consistency of the wall structure $\mathscr{S}$ in codimension zero and one in this modifed sense, Proposition 2.16 on the construction and properties of $\mathfrak{X}^\circ$ then hold true. Moreover, there is still a notion of consistency in codimension two which ensures the existence of enough local functions. One can then proceed to construct the canonical basis ϑ_m of the ring of global functions of $\mathfrak{X}^\circ$ via broken lines as in Section 3. Note however, that now there possibly is an additional dependence of the initial coefficient a_1 of a broken line on the initial maximal cell.

The construction of the partial compactification $\mathfrak{X}$ of $\mathfrak{X}^\circ$ in Section 4 depended on the fact that we can lift the construction to the cone $\mathbf{C}B$ over B. In the present situation this means we need a lift

$$\tilde{\chi}_{\mathfrak{b},\mathfrak{u}} = \chi_{\mathbf{C}\mathfrak{b},\mathbf{C}\mathfrak{u}} : R_{\mathbf{C}\mathfrak{b}} \longrightarrow R_{\mathbf{C}\mathfrak{u}}$$

of $\chi_{\mathfrak{b},\mathfrak{u}}$. Unlike in the untwisted situation this does not follow canonically and is an additional datum to be provided along with $\mathscr{S}$.

To go any further we need to make closer contact with the affine geometry. This is the content of the next subsection.

5.2. **Twisting by gluing data.** We now restrict to the following class of transition maps $\chi_{\mathfrak{b},\mathfrak{u}}$ that covers all cases which have occured in practice so far. Denote by $\chi^{\mathrm{can}}_{\mathfrak{b},\mathfrak{u}}$ the canonical localization homomorphisms of (2.18).

Open gluing data. For any $\underline{\rho} \in \tilde{\mathscr{P}}^{[n-1]}_{\mathrm{int}}$ and adjacent maximal cell σ choose a homomorphism of abelian groups

$$s_{\sigma\underline{\rho}} : \Lambda_\sigma \to A^\times,$$

subject to the constraint

$$s_{\sigma\underline{\rho}}|_{\Lambda_\rho} \cdot \big(s_{\sigma\underline{\rho}'}|_{\Lambda_\rho}\big)^{-1} = s_{\sigma'\underline{\rho}}|_{\Lambda_\rho} \cdot \big(s_{\sigma'\underline{\rho}'}|_{\Lambda_\rho}\big)^{-1}, \tag{5.1}$$

as homomorphisms $\Lambda_\rho \to A^\times$ whenever $\underline{\rho}, \underline{\rho}'$ are contained in the same codimension one cell $\rho \in \mathscr{P}$ with adjacent maximal cells σ and σ'. Following [GrSi2][GrSi4] we call the collection $\mathbf{s} = (s_{\sigma\underline{\rho}})$ *(open) gluing data.* "Open" refers to the fact that these gluing data modify open embeddings, while "closed gluing data" to be considered below can be interpreted as changing the closed embeddings defined by the inclusion of toric strata on X_0. Condition (5.1) is a necessary and sufficient condition to guarantee that the analogue of X_0° exists.

Changing $\chi_{\mathfrak{b},\mathfrak{u}}$. Define the localization homomorphism $\chi_{\mathfrak{b},\mathfrak{u}}$ modified by open gluing data by composing $\chi^{\mathrm{can}}_{\mathfrak{b},\mathfrak{u}} : R_{\mathfrak{b}} \to R_{\mathfrak{u}}$ with the map[12]

$$s_{\sigma\underline{\rho}} : R_{\mathfrak{u}} = (A[Q]/I)[\Lambda_\sigma] \longrightarrow R_{\mathfrak{u}}, \quad z^m \longmapsto s_{\sigma\underline{\rho}}(m) z^m. \tag{5.2}$$

Since now consistency of a wall structure $\mathscr{S}$ in codimension one and two depends on the choice of gluing data we speak of *consistency for the gluing data* $\mathbf{s}$.

Remark 5.1. With trivial gluing data, (2.10) assured that for any $\rho \in \mathscr{P}^{[n-1]}$ all the rings $R_{\underline{\rho}}$ for $\underline{\rho} \subseteq \rho$ are canonically isomorphic. While this statement is superseded by consistency in codimension one (Remark 2.12,5) and hence ultimately is redundant, the local models determine a log structure on X_0° that sometimes is important information. In fact, by [GrSi2], Theorem 3.27, the log structure on X_0° is equivalent to giving functions $f_{\underline{\rho}} \in (A[Q]/I_0)[\Lambda_\rho]$ fulfilling an equation of the form (2.10). The generalization to non-trivial gluing data can be derived from consistency in codimension one and no walls of codimension zero present. To this end consider $\underline{\rho}$, $\underline{\rho}' \subseteq \rho$ with slab functions $f_{\underline{\rho}}$, $f_{\underline{\rho}'}$. Then we have the two models $\operatorname{Spec} R_{\underline{\rho}}$ and $\operatorname{Spec} R_{\underline{\rho}'}$ from (2.11) for the affine neighbourhood in $\mathfrak{X}^\circ$ of the $(n-1)$-stratum $\operatorname{Spec}(A[Q]/I_0)[\Lambda_\rho]$ of X_0°. Requiring these models to be compatible with respect to the localization morphisms twisted by gluing data leads to the following conditions.

First, consistency of gluing of monomials with exponents in Λ_ρ is equivalent to (5.1). Second, for consistency of gluings of $Z_\pm$, let $\xi = \xi(\rho) \in \Lambda_\sigma$ and denote by $Z_\pm$ and $Z'_\pm$ the associated generators of the rings $R_{\underline{\rho}}$ and $R_{\underline{\rho}'}$, respectively. Write also $\xi' \in \Lambda_{\sigma'}$ for $-\xi$ via parallel transport through $\underline{\rho}$ into σ', so that (2.12) holds along $\underline{\rho}'$. Then going from $\underline{\rho}$ to $\underline{\rho}'$ via σ maps Z_+ to $s_{\sigma\underline{\rho}}(\xi)s^{-1}_{\sigma\underline{\rho}'}(\xi) \cdot Z'_+$. Similarly, going via σ' maps Z_- to $s_{\sigma'\underline{\rho}}(\xi')s^{-1}_{\sigma'\underline{\rho}'}(\xi') \cdot Z'_- z^{m_{\underline{\rho}\underline{\rho}'}}$. Note the additional term $m_{\underline{\rho}\underline{\rho}'}$ coming from monodromy. Comparing with the respective equations $Z_+Z_- = f_{\underline{\rho}} z^{\kappa_{\underline{\rho}}}$, $Z'_+Z'_- = f_{\underline{\rho}'} z^{\kappa_{\underline{\rho}'}}$, and assuming (5.1) now leads to the following analogue of (2.10):

$$f_{\underline{\rho}'} z^{\kappa_{\underline{\rho}'}} = \frac{s_{\sigma\underline{\rho}'}(\xi)s_{\sigma'\underline{\rho}'}(\xi')}{s_{\sigma\underline{\rho}}(\xi)s_{\sigma'\underline{\rho}}(\xi')} s^{-1}_{\sigma\underline{\rho}'}(s_{\sigma\underline{\rho}}(f_{\underline{\rho}})) z^{\kappa_{\underline{\rho}}} z^{m_{\underline{\rho}'\underline{\rho}}} \tag{5.3}$$

This equation holding modulo I_0 is necessary and sufficient for the induced log structure on X_0 to glue consistently locally. In particular, it is a necessary condition for the existence of $\mathfrak{X}^\circ$ also under the presence of additional walls and refinements of slabs.

[12]Our sign convention for gluing data in this paper is opposite to the conventions in the previous work of the first and last authors [GrSi2], [GrSi4]. The signs in these works were initially chosen from the point of view of gluing toric strata, but it now is clear that the point of view of gluing open sets is more important.

The choice of gluing data can be formulated cohomologically as follows. Consider the open cover $\underline{\mathscr{W}}$ of B consisting of the open stars of the barycentric subdivision. The notation is $W_{\underline{\tau}}$ for the open star of $\underline{\tau} \in \tilde{\mathscr{P}}$. We also use the notation W_τ for $\tau \in \mathscr{P}$ to denote the open star of $\operatorname{Int}\tau$ with respect to $\tilde{\mathscr{P}}$. Denote by $\underline{\mathscr{W}}_0 \subseteq \underline{\mathscr{W}}$ the subset consisting of interiors of maximal cells σ (the open star of the barycenter of σ) and of the open stars of $\underline{\rho} \in \tilde{\mathscr{P}}^{[n-1]}_{\mathrm{int}}$ not contained in ∂B. Thus the elements of this covering are $W_\sigma = \operatorname{Int}\sigma$ and pairwise disjoint open neighbourhoods $W_{\underline{\rho}}$, one for each $\underline{\rho}$ not contained in ∂B. Since the elements of $\underline{\mathscr{W}}_0$ and their non-trivial intersections $W_{\underline{\rho}\sigma} := W_{\underline{\rho}} \cap W_\sigma$ are contractible, $\underline{\mathscr{W}}_0$ is a Leray covering for the locally constant sheaf $\check{\Lambda} \otimes_{\mathbb{Z}} \underline{A}^\times$ on $B_0 \setminus \partial B$. Moreover, one has $\Gamma(W_{\underline{\rho}\sigma}, \check{\Lambda} \otimes \underline{A}^\times) = \operatorname{Hom}(\Lambda_\sigma, A^\times)$. Thus $(s_{\sigma\underline{\rho}})_{\underline{\rho},\sigma}$ defines a Čech 1-cocycle with values in $\check{\Lambda} \otimes_{\mathbb{Z}} \underline{A}^\times$ for the covering $\underline{\mathscr{W}}_0$, but not all Čech 1-cocycles satisfy (5.1). Cohomologous cocycles lead to isomorphisms between constructions of $\mathfrak{X}^\circ$. To state this note that each pair $(\mathscr{S}, \mathbf{s})$ consisting of a wall structure and gluing data consistent in codimensions zero and one gives rise to a directed system of rings $(R_{\mathfrak{u}}, R_{\mathfrak{b}})$. We are interested in isomorphisms of such associated directed systems of rings acting trivially on the labelling set $\{\mathfrak{u}, \mathfrak{b}\}$. We call such isomorphisms *special*.

Proposition 5.2. *Let $(B, \mathscr{P})$ be a polyhedral pseudomanifold. There is an action of the group $C^0(\underline{\mathscr{W}}_0, \check{\Lambda} \otimes_{\mathbb{Z}} \underline{A}^\times)$ on the set of pairs $(\mathscr{S}, \mathbf{s})$ consisting of a wall structure and open gluing data. This action takes structures consistent in codimension zero and one to structures which are consistent in codimension zero and one. For these consistent structures, the associated directed systems of rings $(R_{\mathfrak{u}}, R_{\mathfrak{b}})$ are related by special isomorphisms.*

Proof. The action of $\mathbf{t} = (t_\sigma, t_{\underline{\rho}}) \in C^0(\underline{\mathscr{W}}_0, \check{\Lambda} \otimes_{\mathbb{Z}} \underline{A}^\times)$ on $\mathbf{s}$ is

$$s_{\sigma\underline{\rho}} \longmapsto t_\sigma \cdot s_{\sigma\underline{\rho}} \cdot t_{\underline{\rho}}^{-1}.$$

Now t_σ acts on the rings R_σ via $z^m \mapsto t_\sigma(m) \cdot z^m$, hence also on the rings $R_{\mathfrak{u}}$ and on the functions $f_{\mathfrak{p}}$ carried by walls. Similarly, $t_{\underline{\rho}}$ acts on the rings $(A[Q]/I)[\Lambda_\rho]$. For a slab $\mathfrak{b}$ define $\mathfrak{b}(\mathbf{t})$ by applying this action to $f_{\mathfrak{b}}$. Then $t_{\underline{\rho}}$ induces an isomorphism $R_{\mathfrak{b}} \to R_{\mathfrak{b}(\mathbf{t})}$ for any slab $\mathfrak{b} \subseteq \rho$, taking $Z_\pm$ to $t_{\underline{\rho}}(\pm\xi) Z_\pm$, where as usual ξ is the chosen element of Λ_x for $x \in \mathfrak{b}$ determining Z_+. The data $\mathbf{t}$ then modifies $\mathscr{S}$ by replacing each slab $\mathfrak{b}$ with the slab $\mathfrak{b}(\mathbf{t})$ and applying t_σ to each wall function $f_{\mathfrak{p}}$ contained in σ. It is then easy to see that this new structure $\mathscr{S}(\mathbf{t})$ is consistent in codimension zero and one with respect to the twisted gluing data, assuming $\mathscr{S}$ was consistent in these codimensions with respect to the original gluing data. Indeed, codimension zero follows trivially. Codimension one consistency follows easily from the definition and the fact that if θ, θ' are the wall-crossing automorphisms occuring in Definition 2.14 for the structure $\mathscr{S}$ and $\theta_{\mathfrak{t}}$, $\theta'_{\mathfrak{t}}$ the corresponding automorphisms for $\mathscr{S}(\mathfrak{t})$, one has $t_\sigma \circ \theta = \theta_{\mathfrak{t}} \circ t_\sigma$ and $t_{\sigma'} \circ \theta' = \theta'_{\mathfrak{t}} \circ t_{\sigma'}$. It is then straightforward to check that the action of t_σ on the rings $R_{\mathfrak{u}}$ with $\mathfrak{u} \subseteq \sigma$ and the action $t_{\underline{\rho}} : R_{\mathfrak{b}} \to R_{\mathfrak{b}(\underline{t})}$ defines a special isomorphism of directed systems of rings. □

In particular, it makes sense to call two sets of open gluing data *equivalent* if they are cohomologous as Čech 1-cocycles.

Closed gluing data. Since our gluing data already changes the gluing modulo I_0, consistency in codimension one and two may fail modulo I_0. Thus we may not even obtain a scheme X_0° over $\mathrm{Spec}(A[Q]/I_0)$. If consistency holds in codimension one, we do obtain $\mathfrak{X}^\circ$, but have no guarantee that there is a scheme X_0 analogous to that of §2.1 containing the reduction X_0° of $\mathfrak{X}^\circ$ modulo I_0 as a dense open subscheme. In general, arbitrary choices of X_0 can be described by *closed gluing data*, which explains how to assemble X_0 by gluing along closed strata. This was carried out in [GrSi2], §2.1. Furthermore, without access to local models in codimension ≥ 2, we will rely on projectivity to compactify $\mathfrak{X}^\circ$, already modulo I_0. We will now explore what additional conditions must be imposed on open gluing data to guarantee the existence of X_0.

There are some obvious obstructions to the existence of X_0 in codimensions one and two associated with interior joints, as follows. Let $\mathfrak{j}$ be an interior joint for the wall structure $\mathscr{S}$ and $\tau = \sigma_{\mathfrak{j}} \in \mathscr{P}$ the minimal cell containing $\mathfrak{j}$. Then for any $m \in \Lambda_\tau$ we have a monomial z^m in the rings $R_{\mathfrak{b}}$ and $R_{\mathfrak{u}}$ for slabs $\mathfrak{b} \supseteq \mathfrak{j}$ and chambers $\mathfrak{u} \supseteq \mathfrak{j}$, but passing from $\mathfrak{b} \subseteq \underline{\rho}$ to an adjacent $\mathfrak{u} \subseteq \sigma$ introduces the factor $s_{\sigma\underline{\rho}}(m)$. Thus $z^m \in R_{\mathfrak{u}}$ extends to a function on the scheme $X_{\mathfrak{j},0}$ of §2.1 constructed from $B_{\mathfrak{j}}$ only if

$$\prod_{i=1}^{r} s_{\sigma_{i+1}\underline{\rho}_i}(m) \cdot s^{-1}_{\sigma_i\underline{\rho}_i}(m) = 1. \tag{5.4}$$

Here the σ_i and $\underline{\rho}_i$ are the maximal and codimension one cells containing $\mathfrak{j}$ ordered in such a way that $\underline{\rho}_i \subseteq \sigma_i \cap \sigma_{i+1}$ and with i taken modulo r. We note that for $\mathfrak{j}$ a joint intersecting the interior of a codimension one cell, the above equation is precisely (5.1) which is assumed to hold for open gluing data. The condition for joints contained in codimension two cells is more subtle. As we will see in Theorem 5.8, (5.1) will be implied by the existence of a version $\bar{s}_{\tau\omega}$ of the closed gluing data for any inclusion of cells $\omega \subseteq \tau$, which twists the construction of X_0 and acts on the starting monomials of broken lines. In the case B is a manifold (with boundary), in fact (5.4) is equivalent to the existence of such closed gluing data. More generally, there is a local cohomology obstruction to the existence of such closed gluing data, see Proposition 5.7 below.

The collection of $\bar{s}_{\tau\omega}$ is a one-cocycle on B for a sheaf $\mathcal{Q}$ which is constructible with respect to a decomposition $\check{\mathscr{P}}$ of B that is dual to $\mathscr{P}$. It is canonically defined by taking the cell $\check{\tau} \in \check{\mathscr{P}}$ dual to $\tau \in \mathscr{P}$ as the union of all cells of the barycentric subdivision labelled by $\tau = \tau_0 \subseteq \tau_1 \subseteq \ldots \subseteq \tau_k$. We will need the following facts about $\check{\mathscr{P}}$ and the open cover $\underline{\mathscr{W}}$, which require a little bit of care because of unbounded cells:

Lemma 5.3. *Suppose given a polyhedral pseudomanifold B of dimension n with decomposition $\mathscr{P}$. We have:*

(1) *If $\tau \in \mathscr{P}$ is unbounded, then $\check{\tau}$ is empty; otherwise $\check{\tau}$ is non-empty, $\dim \check{\tau} = n - \dim \tau$ and $\tau \cap \check{\tau}$ consists of a single point, the barycenter of τ.*

(2) *If $\tau \cap \check{\omega}$ is non-empty, then $\omega \subseteq \tau$ and $\dim \tau \cap \check{\omega} = \dim \tau - \dim \omega$.*

(3) *Let $p \in B$ and let $\underline{\tau}$ be the minimal cell of the barycentric subdivision $\tilde{\mathscr{P}}$ of $\mathscr{P}$ containing p. Suppose that $\underline{\tau}$ corresponds to a sequence of cells $\tau_0 \subseteq \cdots \subseteq \tau_k$. Then there is a sequence of continuous maps $\kappa_i : W_{\underline{\tau}} \to W_{\underline{\tau}}$ compatible with $\tilde{\mathscr{P}}$, with $\kappa_i(p) = p$, κ_i a homeomorphism onto its image, and $\bigcap_i \operatorname{im}(\kappa_i) = \{p\}$. Similarly, there exists a sequence of maps $\kappa_i : W_{\underline{\tau}} \to W_{\underline{\tau}}$ compatible with $\tilde{\mathscr{P}}$, with $\kappa_i|_{\check{\tau}_0}$ the identity, κ_i a homeomorphism onto its image, and $\bigcap_i \operatorname{im}(\kappa_i) = \check{\tau}_0$.*

Proof. (1) In the definition of $\tilde{\mathscr{P}}$, there is no cell of $\tilde{\mathscr{P}}$ corresponding to a chain $\tau_0 \subseteq \cdots \subseteq \tau_k$ with τ_0 (and hence all τ_i) unbounded, hence the first statement. If τ is bounded, it is immediate from the definition that $\tau \cap \check{\tau}$ consists just of the barycenter of τ. To see the dimension statement, choose a chain $\tau = \tau_0 \subseteq \cdots \subseteq \tau_k$ which is maximal, and such that $\tau_0, \ldots, \tau_\ell$ are bounded and $\tau_{\ell+1}, \ldots, \tau_k$ are unbounded. Furthermore, choose these so that ℓ is as large as possible. One can then check that $u_{\tau_{\ell+1}}, \ldots, u_{\tau_k}$ are linearly independent and thus from the definition the corresponding cell of $\tilde{\mathscr{P}}$ is of dimension $n - \dim \tau$. Further, it is clear from the definition that every cell of $\tilde{\mathscr{P}}$ contained in $\check{\tau}$ is dimension at most $n - \dim \tau$, hence the claim.

(2) The first statement follows immediately from the definition of $\check{\omega}$. For the dimension statement, note that the intersection is a union of cells of $\tilde{\mathscr{P}}$ corresponding to chains $\omega = \omega_0 \subseteq \cdots \subseteq \omega_k = \tau$. Such a cell is always of dimension at most $\dim \tau - \dim \omega$, and a similar argument as in (1) shows that there is at least one such cell achieving this bound.

(3) In each of the two cases, it is sufficient to construct maps κ_i defined on each cell $\underline{\omega}$ of $\tilde{\mathscr{P}}$ containing $\underline{\tau}$ which are compatible with inclusions of faces. To

this end, suppose $\underline{\omega}$ corresponds to a sequence of cells $\omega_0 \subseteq \cdots \subseteq \omega_m$ of $\mathscr{P}$, with $\omega_0, \ldots, \omega_\ell$ bounded and $\omega_{\ell+1}, \ldots, \omega_m$ unbounded. The condition $\underline{\tau} \subseteq \underline{\omega}$ is equivalent to the sequence $\tau_0, \ldots, \tau_k$ being a subsequence of $\omega_0, \ldots, \omega_m$. Recall that

$$\underline{\omega} = \mathrm{conv}\{a_{\omega_0}, \ldots, a_{\omega_\ell}\} + \sum_{\ell+1 \le j \le p} \mathbb{R}_{\ge 0} u_{\omega_j}.$$

By passing to a subsequence of cells, we can assume that the vectors $u_{\ell+1}, \ldots, u_m$ are linearly independent without changing $\underline{\omega}$. Then every element of $\underline{\omega}$ has a unique representative as $\sum \alpha_j a_{\omega_j} + \sum \alpha_j u_{\omega_j}$, with $\sum_{j=0}^{\ell} \alpha_j = 1$. In particular, for the first case, we write p in this form, with $\alpha_j = 0$ if ω_j does not appear in the sequence $\{\tau_j\}$.

Choose once and for all a sequence of maps $\phi_i : \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ such that ϕ_i is a homeomorphism onto its image, $\phi_i(0) = 0$, and $\bigcap_i \mathrm{im}(\phi_i) = \{0\}$. Also fix a sequence of real numbers $\lambda_i \in (0, 1]$ with $\lambda_i \to 0$. Define $\psi_{ij} : [0, 1] \to [0, 1]$ for $0 \le j \le \ell$ by

$$\psi_{ij}(\beta) = \lambda_j(\beta - \alpha_j) + \alpha_j$$

and $\psi_{ij} : \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ for $\ell + 1 \le j \le m$ by

$$\psi_{ij}(\beta) = \begin{cases} \phi_j(\beta - \alpha_j) + \alpha_j & \beta \ge \alpha_j \\ \lambda_j(\beta - \alpha_j) + \alpha_j & \beta \le \alpha_j. \end{cases}$$

Then define $\kappa_i : \underline{\omega} \to \underline{\omega}$ by

$$\kappa_i\left(\sum_{j=0}^{\ell} \beta_j a_{\omega_j} + \sum_{j=\ell+1}^{m} \beta_j u_{\omega_j}\right) = \sum_{j=0}^{\ell} \psi_{ij}(\beta_j) a_{\omega_j} + \sum_{j=\ell+1}^{m} \psi_{ij}(\beta_j) u_{\omega_j}.$$

For the sequence of maps with image converging to $\check{\tau}_0$, suppose $\tau_0 = \omega_q$. Necessarily τ_0 is bounded, so $q \le \ell$. We instead define

$$\begin{aligned} &\kappa_i\left(\sum_{j=0}^{\ell} \beta_j a_{\omega_j} + \sum_{j=\ell+1}^{m} \beta_j u_{\omega_j}\right) \\ &= \sum_{j=0}^{q-1} \lambda_i \beta_j a_{\omega_j} + \sum_{j=q}^{\ell} \left(\frac{1 - \lambda_i \sum_{h=0}^{q-1} \beta_h}{\sum_{h=q}^{\ell} \beta_h}\right) \beta_j a_{\omega_j} + \sum_{j=\ell+1}^{m} \lambda_i \beta_j u_{\omega_j}. \end{aligned}$$

One checks easily that these maps are homeomorphism onto their images and $\bigcap_i \mathrm{im}(\kappa_i) = \check{\tau}_0$. □

The sheaf $\mathcal{Q}$ is defined as the sheaf constructible with respect to $\check{\mathscr{P}}$ with constant stalks

$$\mathcal{Q}_{\check{\tau}} := \check{\Lambda}_\tau = \mathrm{Hom}(\Lambda_\tau, \mathbb{Z})$$

along $\operatorname{Int}\check{\tau}$. For $\check{\tau} \subseteq \check{\omega}$ the generization map $\mathcal{Q}_{\check{\tau}} \to \mathcal{Q}_{\check{\omega}}$ is defined as the dual of the inclusion $\Lambda_\omega \to \Lambda_\tau$. If $\omega \subseteq \tau$ then $\check{\Lambda}_\omega$ surjects onto each stalk $\mathcal{Q}_x$ for $x \in W_\omega \cap W_\tau$ and hence

$$\Gamma(W_\omega \cap W_\tau, \mathcal{Q}) = \check{\Lambda}_\omega.$$

Thus a one-cocycle for $\mathcal{Q} \otimes_{\mathbb{Z}} A^\times$ with respect to the open cover

$$\mathscr{W} = \{W_\tau \,|\, \tau \in \mathscr{P}\}$$

is a collection of homomorphisms $\bar{s}_{\tau\omega} : \Lambda_\omega \to A^\times$, one for each $\omega \subseteq \tau$, fulfilling the cocycle condition $\bar{s}_{\tau''\tau'}|_{\Lambda_\tau} \cdot \bar{s}_{\tau'\tau} = \bar{s}_{\tau''\tau}$. As in the case with GS-type singularities [GrSi2], Definition 2.10, we refer to these one-cocycles as *closed gluing data*, with a notion of equivalence defined by coboundaries. Note also that the same argument as in [GrSi2], Lemma 5.5 shows that $\mathscr{W}$ is an acyclic cover for $\mathcal{Q} \otimes \underline{A}^\times$ and hence

$$H^k(B, \mathcal{Q} \otimes \underline{A}^\times) = H^k(\mathscr{W}, \mathcal{Q} \otimes \underline{A}^\times).$$

In what follows, we denote for $n \geq 2$ the subset $\Delta_n \subseteq \Delta$ consisting of the codimension n skeleton of $\mathscr{P}$. Note that $\Delta \setminus \Delta_2$ is covered by the interiors of those $(n-2)$-cells of $\tilde{\mathscr{P}}$ intersecting the interiors of $(n-1)$-cells of $\mathscr{P}$. We also make use of the open cover of $B \setminus \Delta_2$ given by

$$\mathscr{W}_1 = \{W_\tau \,|\, \tau \in \mathscr{P}, \dim \tau \text{ is } n \text{ or } n-1\}.$$

We then have

$$H^k(B \setminus \Delta_2, \mathcal{Q} \otimes \underline{A}^\times) = H^k(\mathscr{W}_1, \mathcal{Q} \otimes \underline{A}^\times).$$

Lemma 5.4. *Let* $\mathbf{s}$ *be open gluing data for* $(B, \mathscr{P}, \varphi)$*. Then* $\mathbf{s}$ *uniquely determines an element of* $H^1(\mathscr{W}_1, \mathcal{Q} \otimes \underline{A}^\times)$*.*

Proof. For each codimension one $\rho \in \mathscr{P}$, $\rho \not\subseteq \partial B$, choose an ordering σ, σ' of the two maximal cells containing ρ. Replace $\mathbf{s}$ with the cohomologous cycle using the action of Proposition 5.2 with $t_{\underline{\rho}_i} = s_{\sigma'\underline{\rho}_i}$ and $t_\sigma = t_{\sigma'} = 1$. Thus we can assume that $s_{\sigma'\underline{\rho}_i}(m) = 1$ for all m. Now let $\underline{\rho}_1, \underline{\rho}_2$ be two codimension one cells of the barycentric subdivision contained in a common codimension one cell ρ of $\mathscr{P}$, and contained in two codimension zero cells σ, σ'. Then (5.1) now simply states that

$$s_{\sigma\underline{\rho}_1}|_{\Lambda_\rho} = s_{\sigma\underline{\rho}_2}|_{\Lambda_\rho},$$

while the same statement for σ' is trivially true since each side of the equality is 1 by construction. Thus defining $\bar{s}_{\sigma\rho} = s_{\sigma\underline{\rho}}|_{\Lambda_\rho}$ and $\bar{s}_{\sigma'\rho} = 1$, $\underline{\rho} \subseteq \rho$, we obtain well-defined sections of $\mathcal{Q} \otimes \underline{A}^\times$ over $W_\sigma \cap W_\rho$ and $W_{\sigma'} \cap W_\rho$ respectively. So we obtain a Čech one-cocycle $\bar{\mathbf{s}} = (\bar{s}_{\sigma\rho})$ for the sheaf $\mathcal{Q} \otimes \underline{A}^\times$ on the cover $\mathscr{W}_1$. One checks easily that if $\mathbf{s}$ is replaced by a cohomologous cycle then $\bar{\mathbf{s}}$ is also replaced by a cohomologous cycle, hence giving a well-defined element of $H^1(\mathscr{W}_1, \mathcal{Q} \otimes \underline{A}^\times)$ as desired. □

Definition 5.5. If $\mathbf{s}$ is open gluing data for $(B,\mathscr{P},\varphi)$ we write $\bar{\mathbf{s}}$ for the induced element of $H^1(B\setminus\Delta_2,\mathcal{Q}\otimes\underline{A}^\times)$.

We now define an obstruction class $\mathrm{ob}_{\Delta_2}(\bar{\mathbf{s}})$ which is the obstruction to extending $\bar{\mathbf{s}}$ from $B\setminus\Delta_2$ to B, defined via the connecting homomorphism in the long exact sequence for local cohomology:

(5.5)

$$H^1_{\Delta_2}(B,\mathcal{Q}\otimes\underline{A}^\times)\longrightarrow H^1(B,\mathcal{Q}\otimes\underline{A}^\times)\longrightarrow H^1(B\setminus\Delta_2,\mathcal{Q}\otimes\underline{A}^\times)\xrightarrow{\mathrm{ob}_{\Delta_2}} H^2_{\Delta_2}(B,\mathcal{Q}\otimes\underline{A}^\times).$$

Lemma 5.6. *The local cohomology sheaves $\mathcal{H}^k_{\Delta_2}(\mathcal{Q})$ vanish for $k=0,1$. In particular, $H^k_{\Delta_2}(\mathcal{Q})=0$ for $k=0,1$,*

$$\mathcal{H}^k_{\Delta_2}(\mathcal{Q})=R^{k-1}j_*(\mathcal{Q}|_{B_0}),\quad k\ge 2,$$

with $j:B_0\to B$ the inclusion, and $H^2_{\Delta_2}(B,\mathcal{Q})=H^0\big(B,\mathcal{H}^2_{\Delta_2}(\mathcal{Q})\big)$. Analogous statements hold for $\mathcal{Q}\otimes\underline{A}^\times$.

Proof. The first two cohomology sheaves with closed support fit into the exact sequence

$$0\longrightarrow\mathcal{H}^0_{\Delta_2}(\mathcal{Q})\longrightarrow\mathcal{Q}\longrightarrow j_*(\mathcal{Q}|_{B\setminus\Delta_2})\longrightarrow\mathcal{H}^1_{\Delta_2}(\mathcal{Q})\longrightarrow 0,$$

while $\mathcal{H}^k_{\Delta_2}(\mathcal{Q})=R^{k-1}j_*(\mathcal{Q}|_{B_0})$ for $k\ge 2$, see [Hr1], Corollary 1.9. For the vanishing of the first two local cohomology sheaves we thus have to show that $\mathcal{Q}\to j_*(\mathcal{Q}|_{B\setminus\Delta_2})$ is an isomorphism. Let $p\in\Delta_2$ and let $\underline{\tau}$ be the minimal cell of the barycentric decomposition $\tilde{\mathscr{P}}$ containing p. Recall that $\mathcal{Q}$ is a constructible sheaf for $\check{\mathscr{P}}$, hence also for the refinement $\tilde{\mathscr{P}}$. By Lemma 5.3, (3), there is a sequence of retractions $\kappa_i:W_{\underline{\tau}}\to W_{\underline{\tau}}$ compatible with $\tilde{\mathscr{P}}$ and with $\kappa_i^*\mathcal{Q}\simeq\mathcal{Q}$ and $\bigcap_i\mathrm{im}(\kappa_i)=\{p\}$. This shows that $(j_*\mathcal{Q})_p=H^0(W_{\underline{\tau}}\setminus\Delta_2,\mathcal{Q})$.

On the other hand, if $\underline{\tau}$ corresponds to a sequence $\tau_0\subsetneq\cdots\subsetneq\tau_k$ of cells of $\mathscr{P}$, then using Lemma 5.3, (3), a similar retraction argument shows that $H^0(W_{\underline{\tau}}\setminus\Delta_2,\mathcal{Q})\cong H^0((W_{\underline{\tau}}\cap\check{\tau}_0)\setminus\Delta_2,\mathcal{Q}|_{(W_{\underline{\tau}}\cap\check{\tau}_0)\setminus\Delta_2})$. Note $\mathcal{Q}|_{W_{\underline{\tau}}\cap\check{\tau}_0}$ is a constant sheaf, and thus $H^0(W_{\underline{\tau}},\mathcal{Q})=H^0(W_{\underline{\tau}}\cap\check{\tau}_0,\mathcal{Q}|_{\check{\tau}_0})=\check{\Lambda}_{\tau_0}$.

It follows from the S_2 condition on B that $W_{\underline{\tau}}\setminus\Delta_2$ is connected. Indeed, this can be shown inductively by computing $H^0(W_{\underline{\tau}}\setminus\Delta_i,\underline{\mathbb{Z}})$ for $i=\operatorname{codim}\tau_k,\ldots,2$, and observing that the S_2 condition implies that

$$H^0_{\Delta_i\setminus\Delta_{i+1}}(W_{\underline{\tau}}\setminus\Delta_{i+1},\underline{\mathbb{Z}})=H^1_{\Delta_i\setminus\Delta_{i+1}}(W_{\underline{\tau}}\setminus\Delta_{i+1},\underline{\mathbb{Z}})=0.$$

Since $W_{\underline{\tau}}\setminus\Delta_2$ retracts onto $(W_{\underline{\tau}}\cap\check{\tau}_0)\setminus\Delta_2$, it follows that the latter is connected and thus $H^0((W_{\underline{\tau}}\cap\check{\tau}_0)\setminus\Delta_2,\mathcal{Q}|_{(W_{\underline{\tau}}\cap\check{\tau}_0)\setminus\Delta_2})=\check{\Lambda}_{\tau_0}$. We conclude that the map $\mathcal{Q}\to j_*(\mathcal{Q}|_{B\setminus\Delta_2})$ is an isomorphism, as desired.

The claims on $H^k_{\Delta_2}(B,\mathcal{Q})$, $k\le 2$, now follow from the local to global spectral sequence for cohomology with supports [Hr1], Proposition 1.4. □

Proposition 5.7. *A one-cocycle* $\bar{\mathbf{s}} = (\bar{s}_{\sigma\rho})$ *for* $\mathcal{Q}\otimes\underline{A}^\times$ *on* $B\setminus\Delta_2$ *extends to a one-cocycle on* B *if and only if the local obstruction* $\mathrm{ob}_{\Delta_2}(\bar{\mathbf{s}}) \in \Gamma\big(B, \mathcal{H}^2_{\Delta_2}(\mathcal{Q}\otimes\underline{A}^\times)\big)$ *for doing so vanishes. An extension is unique up to equivalence.*

Proof. The existence statement is immediate from (5.5) and Lemma 5.6. The same sequence shows that the equivalence class of the extension is unique up to the action of $H^1_{\Delta_2}(\mathcal{Q}\otimes\underline{A}^\times)$, which vanishes by Lemma 5.6. □

We will now connect the vanishing of the local obstruction class $\mathrm{ob}_{\Delta_2}(\bar{\mathbf{s}})$ with (5.4), obtaining the strongest results in the case that B is topological manifold with boundary.

Proposition 5.8. *Let* $\mathbf{s}$ *be open gluing data for* $(B,\mathscr{P},\varphi)$*. If the obstruction* $\mathrm{ob}_{\Delta_2}(\bar{\mathbf{s}}) \in H^2_{\Delta_2}(B,\mathcal{Q}\otimes\underline{A}^\times)$ *for extending* $\mathbf{s}$ *to closed gluing data on all of* B *vanishes then the consistency condition* (5.4) *holds for interior joints* $\mathfrak{j}$ *of the form* $\underline{\tau} \in \tilde{\mathscr{P}}$ *contained in* Δ_2 *and for all* $m \in \Lambda_\tau$*,* $\tau \in \mathscr{P}$ *the minimal cell containing* $\underline{\tau}$*. Furthermore, this implication is an equivalence if* B *is a topological manifold with boundary.*

Proof. By Proposition 5.7 it suffices to consider the vanishing statement locally. We first consider the case that B is a topological manifold with boundary, and in this case show that for the vanishing of the obstruction $\mathrm{ob}_{\Delta_2}(\bar{\mathbf{s}}) \in H^2_{\Delta_2}(B,\mathcal{Q}\otimes\underline{A}^\times)$, it is sufficient to consider the vanishing at general points of the codimension two cells covering Δ_2. We will then show that this latter vanishing is in any case equivalent to (5.4).

To this end, suppose B is a manifold with boundary, and consider part of the long exact sequence of cohomology with supports for Δ_3 ([Hr1], Proposition 1.9):

$$\mathcal{H}^2_{\Delta_3}(\mathcal{Q}\otimes\underline{A}^\times) \longrightarrow \mathcal{H}^2_{\Delta_2}(\mathcal{Q}\otimes\underline{A}^\times) \longrightarrow \mathcal{H}^2_{\Delta_2\setminus\Delta_3}(\mathcal{Q}\otimes\underline{A}^\times).$$

We claim that $\mathcal{H}^2_{\Delta_3}(\mathcal{Q}\otimes\underline{A}^\times) = 0$. Then in view of the excision formula ([Hr1], Proposition 1.3) vanishing of the local obstruction class can be tested on $B\setminus\Delta_3$. To prove the claim denote by $j_3 : B\setminus\Delta_3 \to B$ the inclusion. Then $\mathcal{H}^2_{\Delta_3}(\mathcal{Q}\otimes\underline{A}^\times) = R^1 j_{3*}(\mathcal{Q}\otimes\underline{A}^\times)$ ([Hr1], Corollary 1.9). Let $p \in \Delta_3$ and $\underline{\tau}$ the minimal cell of the barycentric decomposition $\tilde{\mathscr{P}}$ containing p. By the same retraction argument of $W_{\underline{\tau}}$ to p as in the proof of Lemma 5.6, we obtain

$$\big(R^1 j_{3*}(\mathcal{Q}\otimes\underline{A}^\times)\big)_p = H^1(W_{\underline{\tau}}\setminus\Delta_3, \mathcal{Q}\otimes\underline{A}^\times).$$

If $\underline{\tau}$ corresponds to a sequence $\tau_0 \subsetneq \cdots \subsetneq \tau_k$ of cells of $\mathscr{P}$, then similarly to the proof of Lemma 5.6, we have

$$H^1(W_{\underline{\tau}}\setminus\Delta_3, \mathcal{Q}\otimes\underline{A}^\times) \cong H^1((W_{\underline{\tau}}\cap\check{\tau}_0)\setminus\Delta_3, (\mathcal{Q}\otimes\underline{A}^\times)|_{W_{\underline{\tau}}\cap\check{\tau}_0}).$$

Furthermore, by Lemma 5.3, (2), it follows that $\Delta_3 \cap \check{\tau}_0$ is codimension three in $\check{\tau}_0$. In particular, since $\mathcal{Q}|_{W_{\underline{\tau}} \cap \check{\tau}_0}$ is a constant sheaf with stalk $\check{\Lambda}_{\tau_0}$ and, if we assume B is a manifold with boundary, $W_{\underline{\tau}} \cap \check{\tau}_0$ is an open ball in a manifold with boundary, we see that

$$H^1((W_{\underline{\tau}} \cap \check{\tau}_0) \setminus \Delta_3, (\mathcal{Q} \otimes \underline{A}^\times)|_{W_{\underline{\tau}} \cap \check{\tau}_0}) = 0. \tag{5.6}$$

13→ This finishes the proof of the claim if B is a manifold with boundary. [13]

Now consider B arbitrary, and consider the map $\mathrm{ob}_{\Delta_2 \setminus \Delta_3} : H^2(B \setminus \Delta_2, \mathcal{Q} \otimes \underline{A}^\times) \to H^2_{\Delta_2 \setminus \Delta_3}(B \setminus \Delta_3, \mathcal{Q} \otimes \underline{A}^\times)$. We have just shown that if B is a manifold with boundary, then $\mathrm{ob}_{\Delta_2 \setminus \Delta_3}(\bar{\mathbf{s}}) = 0$ if and only if $\mathrm{ob}_{\Delta_2}(\bar{\mathbf{s}}) = 0$. We now show in any case that vanishing of $\mathrm{ob}_{\Delta_2 \setminus \Delta_3}(\bar{s})$ is equivalent to the stated consistency condition. We have $H^2_{\Delta_2 \setminus \Delta_3}(B \setminus \Delta_3, \mathcal{Q} \otimes \underline{A}^\times) \cong H^0(\mathcal{H}^2_{\Delta_2 \setminus \Delta_3}(B \setminus \Delta_3, \mathcal{Q} \otimes \underline{A}^\times))$. By constructibility of $\mathcal{Q}$ it suffices to test the vanishing of a section of $\mathcal{H}^2_{\Delta_2 \setminus \Delta_3}(B \setminus \Delta_3, \mathcal{Q} \otimes \underline{A}^\times)$ at p the barycenter $\tau \cap \check{\tau} \in \tilde{\mathscr{P}}^{[0]}$ of a cell $\tau \in \mathscr{P}$ of codimension two. By the same argument of constructibility as in the discussion of the codimension three locus, there is an isomorphism

$$\big(\mathcal{H}^2_{\Delta_2 \setminus \Delta_3}(\mathcal{Q} \otimes \underline{A}^\times)\big)_p = H^1(W_\tau \setminus \Delta_2, \mathcal{Q} \otimes \underline{A}^\times) = H^1(W_\tau \setminus \tau, \mathcal{Q} \otimes \underline{A}^\times).$$

In the present case there is a sequence of retractions $\kappa_k : W_\tau \setminus \tau \to W_\tau \setminus \tau$ with $\bigcap_k \mathrm{im}(\kappa_k) = (\mathrm{Int}\, \check{\tau}) \setminus \{p\}$, and hence

$$H^1(W_\tau \setminus \tau, \mathcal{Q} \otimes \underline{A}^\times) = H^1\big((\mathrm{Int}\, \check{\tau}) \setminus \{p\}, \mathcal{Q} \otimes \underline{A}^\times\big).$$

As before, the restriction of $\mathcal{Q} \otimes \underline{A}^\times$ to $\mathrm{Int}\, \check{\tau}$ is a constant sheaf with stalks $\check{\Lambda}_\tau \otimes \underline{A}^\times$. Hence we can compute

$$\big(\mathcal{H}^2_{\Delta_2 \setminus \Delta_3}(\mathcal{Q} \otimes \underline{A}^\times)\big)_p = H^1\big((\mathrm{Int}\, \check{\tau}) \setminus \{p\}, \check{\Lambda}_\tau \otimes A^\times\big).$$

If $\tau \subseteq \partial B$, then $(\mathrm{Int}\, \check{\tau}) \setminus \{p\}$ is contractible, and this group is zero. Otherwise, $(\mathrm{Int}\, \check{\tau}) \setminus \{p\}$ is homotopic to S^1 and we obtain

$$\big(\mathcal{H}^2_{\Delta_2 \setminus \Delta_3}(\mathcal{Q} \otimes \underline{A}^\times)\big)_p = H^1(S^1, \check{\Lambda}_\tau \otimes \underline{A}^\times) = \check{\Lambda}_\tau \otimes \underline{A}^\times.$$

Under this sequence of identifications the restriction of the obstruction class $\mathrm{ob}_{\Delta_2 \setminus \Delta_3}(\bar{\mathbf{s}})$ is mapped to $\prod_{i=1}^r \bar{s}_{\sigma_{i+1}\rho_i}(m) \cdot \bar{s}^{-1}_{\sigma_i \rho_i}(m)$. Thus the local obstruction vanishes along $\mathrm{Int}\, \tau$ if and only if the consistency condition (5.4) holds for all $m \in \Lambda_\tau$. □

Definition 5.9. We say open gluing data $\mathbf{s}$ for $(B, \mathscr{P}, \varphi)$ is *consistent* if $\mathrm{ob}_{\Delta_2}(\bar{\mathbf{s}}) = 0$. In this case, we obtain uniquely induced closed gluing data $\bar{\mathbf{s}} \in H^1(B, \mathcal{Q} \otimes \underline{A}^\times)$, by Proposition 5.7.

[13] (Mark) Double check whether we have said the right words around bits of $\Delta_2 \cap \partial B$.

Remark 5.10. In fact if $\bar{\mathbf{s}}$ exists for a given $\mathbf{s}$, then we can assume that we have specific representatives of both such that for $\rho \subseteq \sigma \in \mathscr{P}_{\max}$ with ρ codimension one, and any $\underline{\rho} \subseteq \rho$, $m \in \Lambda_\rho$, one has

$$\bar{s}_{\sigma\rho}(m) = s_{\sigma\underline{\rho}}(m). \tag{5.7}$$

Indeed, the argument of the proof of Lemma 5.4 implies we can replace $\mathbf{s}$ with equivalent open gluing data so that we can define a cocycle $\bar{\mathbf{s}}$ for $\mathcal{Q} \otimes A^\times$ over $B \setminus \Delta_2$ by (5.7). The vanishing of $\mathrm{ob}_{\Delta_2}(\bar{\mathbf{s}})$ then implies $\bar{\mathbf{s}}$ lifts as a cohomology class $\bar{\mathbf{s}}'$ to $H^1(B, \mathcal{Q} \otimes \underline{A}^\times)$. Thus the restriction of $\bar{\mathbf{s}}'$ to $B \setminus \Delta_2$ is cohomologous to $\bar{\mathbf{s}}$, that is, there exists a collection of data $t_\sigma \in \check{\Lambda}_\sigma \otimes \underline{A}^\times$, $t_\rho \in \check{\Lambda}_\tau \otimes \underline{A}^\times$ such that

$$\bar{s}'_{\sigma\rho}(m) = t_\sigma(m)\bar{s}_{\sigma\rho}(m)t_\rho(m)^{-1}$$

for all $\rho \subseteq \sigma$, $m \in \Lambda_\rho$. For each $\underline{\rho} \subseteq \rho$, choose a lift $t_{\underline{\rho}}$ of t_ρ to $\Gamma(W_{\underline{\rho}}, \check{\Lambda} \otimes_{\mathbb{Z}} \underline{A}^\times)$. Replacing $\mathbf{s}$ by the equivalent open gluing data induced by the t_σ and $t_{\underline{\rho}}$ then yields open gluing data $\mathbf{s}$ satisfying (5.7).

Let us now assume given consistent open gluing data $\mathbf{s}$. Then we have a unique choice of induced closed gluing data $\bar{\mathbf{s}} = (\bar{s}_{\tau\omega})$. Unfortunately, having induced closed gluing data is insufficient to construct X_0 as a scheme; at best one can hope only to construct X_0 as an algebraic space as a direct limit of closed immersions of toric varieties twisted by closed gluing data. However, we shall take an easier route in the case that X_0 still carries an ample line bundle. This case is detected by another obstruction, which we turn to now.

Projectivity. As we want to follow the strategy from Section 4 for the partial completion of $\mathfrak{X}^\circ$, we need to go over to the cone $\mathbf{C}B$ and construct global functions on the corresponding affine scheme. This process is obstructed in general already on X_0 for there exist non-projective locally trivial deformations of such schemes. An example is provided by certain regluings of the degenerate quartic surface $X_0X_1X_2X_3 = 0$ in $\mathbb{P}^3$, see [Fr], Remark 2.12.

The first problem is the lifting of gluing data $\mathbf{s}$ and $\bar{\mathbf{s}}$ to $\mathbf{C}B$. Let $h : \mathbf{C}B \to \mathbb{R}_{\geq 0}$ be the height function and identify B with the slice $h^{-1}(1) \subseteq \mathbf{C}B$. Denote by $r : \mathbf{C}B \setminus h^{-1}(0) \to B$ the retraction along rays emanating from the apex. Recall that r does not respect the affine structure, but h does. If we denote by $\tilde{\mathcal{Q}}$ the sheaf analogous to $\mathcal{Q}$ on $\mathbf{C}B$, we have the exact sequence on $\mathbf{C}B \setminus h^{-1}(0)$

$$0 \longrightarrow \underline{A}^\times \xrightarrow{h^*} \tilde{\mathcal{Q}} \otimes \underline{A}^\times \longrightarrow r^*\mathcal{Q} \otimes \underline{A}^\times \longrightarrow 0. \tag{5.8}$$

In this sequence the morphism to $r^*\mathcal{Q} \otimes \underline{A}^\times$ is induced by identifying $r^*\Lambda$ with $\ker h_* \subseteq \tilde{\Lambda}$. Then we can view $\bar{\mathbf{s}}$ as an element in $H^1(B, \mathcal{Q} \otimes \underline{A}^\times) \cong H^1(\mathbf{C}B \setminus$

$h^{-1}(0), r^*\mathcal{Q} \otimes \underline{A}^\times)$, and hence we have an element

$$\mathrm{ob}_{\mathbb{P}}(\bar{\mathbf{s}}) \in H^2(\mathbf{C}B \setminus h^{-1}(0), A^\times) \cong H^2(B, A^\times) \tag{5.9}$$

via the connecting homomorphism in the long exact cohomology sequence of (5.8).

Definition 5.11. Fix a Čech representative $(\bar{s}_{\tau\omega})_{\omega\subseteq\tau}$ for closed gluing data $\bar{\mathbf{s}}$. Suppose $\tilde{\bar{\mathbf{s}}}, \tilde{\bar{\mathbf{s}}}'$ are two lifts of $\bar{\mathbf{s}}$ to $H^1(\mathbf{C}B\setminus h^{-1}(0), \tilde{\mathcal{Q}}\otimes A^\times) \cong H^1(B, (\tilde{\mathcal{Q}}\otimes A^\times)|_B)$, given by representatives $(\tilde{\bar{s}}_{\tau\omega})$, $(\tilde{\bar{s}}'_{\tau\omega})$ with the image of both $\tilde{\bar{s}}_{\tau\omega}$ and $\tilde{\bar{s}}'_{\tau\omega}$ in $\check{\Lambda}_\omega \otimes \underline{A}^\times$ coinciding with $\bar{s}_{\tau\omega}$. Then we say $\tilde{\bar{\mathbf{s}}}$ and $\tilde{\bar{\mathbf{s}}}'$ are *equivalent* if there exists for all $\omega \in \mathscr{P}$ a choice of $t_\omega \in A^\times$ such that, viewing t_ω as a section of $\tilde{\mathcal{Q}} \otimes A^\times$ via h^*,

$$\tilde{\bar{s}}'_{\tau\omega} = t_\tau \tilde{\bar{s}}_{\tau\omega} t_\omega^{-1}$$

for all $\omega \subseteq \tau$.

We then have

Proposition 5.12. *For consistent open gluing data* $\mathbf{s}$ *with associated closed gluing data* $\bar{\mathbf{s}}$, $\mathrm{ob}_{\mathbb{P}}(\bar{\mathbf{s}})$ *vanishes if and only if* $\bar{\mathbf{s}}$ *lifts to closed gluing data for* $\mathbf{C}B$. *Moreover, if* $\mathrm{ob}_{\mathbb{P}}(\bar{\mathbf{s}})$ *vanishes, then the set of lifts* $\tilde{\bar{\mathbf{s}}}$ *up to equivalence is a torsor for* $H^1(B, \underline{A}^\times)$. *Finally, for each such lift* $\tilde{\bar{\mathbf{s}}}$, *there is a choice of open gluing data* $\tilde{\mathbf{s}}$ *for* $\mathbf{C}B$ *inducing closed gluing data* $\tilde{\bar{\mathbf{s}}}$.

Proof. It is clear that the first two statements follow from the long exact cohomology sequence for (5.8), using Čech cohomology for the cover $\mathscr{W}$. For the last statement, we can assume $\mathbf{s}$ and $\bar{\mathbf{s}}$ are related as in Remark 5.10. Then given the lift $\tilde{\bar{\mathbf{s}}}$ of $\bar{\mathbf{s}}$, we construct a lift of $\mathbf{s}$ to open gluing data $\tilde{\mathbf{s}}$ for $\mathbf{C}B$ simply by defining, for $m \in \tilde{\Lambda}_{\mathbf{C}\rho}$, $\tilde{s}_{\sigma\underline{\rho}}(m) = \tilde{\bar{s}}_{\sigma\rho}(m)$, and for $m \in \Lambda_\sigma \subseteq \tilde{\Lambda}_{\mathbf{C}\sigma}$, $\tilde{s}_{\sigma\underline{\rho}}(m) = s_{\sigma\underline{\rho}}(m)$. ☐

Proposition 5.12 prompts us to make the following definition.

Definition 5.13. We call consistent open gluing data $\mathbf{s}$ *projective* if the induced closed gluing data $\bar{\mathbf{s}} \in H^1(B, \mathcal{Q} \otimes \underline{A}^\times)$ satisfies $\mathrm{ob}_{\mathbb{P}}(\bar{\mathbf{s}}) = 0$.

Remark 5.14. If the singularities of B are of the type considered in [GrSi2] and the closed gluing data $\mathbf{s}$ are the restriction of open gluing data $\mathbf{s}'$ on all of B, then $\mathrm{ob}_{\mathbb{P}}(\mathbf{s}) \in H^2(B, \Bbbk^\times)$ agrees with the image of $\mathbf{s}'$ under the homomorphism o in [GrSi2], Theorem 2.34.

We are then in position to modify the constructions from Sections 2 and 3 consistently as follows.

Construction of X_0. Assume now given consistent open gluing data $\mathbf{s}$ with induced closed gluing data $\bar{\mathbf{s}}$. Suppose further that $\mathrm{ob}_{\mathbb{P}}(\bar{\mathbf{s}})$ vanishes, and choose a lift $\tilde{\bar{\mathbf{s}}}$ of $\bar{\mathbf{s}}$. In the notation of §2.1 define the ring $S[B](\tilde{\bar{\mathbf{s}}})$ with the same elements as $S[B]$ but with the multiplication of monomials $z^m \cdot z^{m'}$ modified as follows. Let ω, ω' be the minimal cells with $m \in \mathbf{C}\omega$, $m' \in \mathbf{C}\omega'$. Assume that there is a cell τ containing $\omega \cup \omega'$. Taking τ minimal with this property we define

$$z^m \cdot z^{m'} := \tilde{\bar{s}}_{\tau\omega}(m)\tilde{\bar{s}}_{\tau\omega'}(m')z^{m+m'}. \tag{5.10}$$

If no such τ exists the product is zero as before. As for associativity let m, m', m'' be contained in the cones for $\omega, \omega', \omega''$ and assume σ is the minimal cell containing $\omega \cup \omega' \cup \omega''$. Denote by τ, τ', τ'' the minimal cells containing $\omega' \cup \omega'', \omega'' \cup \omega$ and $\omega \cup \omega'$, respectively. Then

$$\begin{aligned}(z^m \cdot z^{m'}) \cdot z^{m''} &= \tilde{\bar{s}}_{\tau''\omega}(m)\tilde{\bar{s}}_{\tau''\omega'}(m')\tilde{\bar{s}}_{\sigma\tau''}(m+m')\tilde{\bar{s}}_{\sigma\omega''}(m'')z^{m+m'+m''}\\ &= \tilde{\bar{s}}_{\sigma\omega}(m)\tilde{\bar{s}}_{\sigma\omega'}(m')\tilde{\bar{s}}_{\sigma\omega''}(m'')z^{m+m'+m''}.\end{aligned}$$

For the second equality we used multiplicativity of $\tilde{\bar{s}}_{\sigma\tau''}$ and the cocycle conditions for $\omega \subseteq \tau'' \subseteq \sigma$ and $\omega' \subseteq \tau'' \subseteq \sigma$.

We now take $X_0 = \operatorname{Proj} S[B](\tilde{\bar{\mathbf{s}}})$. To characterize the irreducible components of the modified X_0 (Proposition 2.2) note that the monomials for a fixed $\sigma \in \mathscr{P}$ generate a subring of $S[B](\tilde{\bar{\mathbf{s}}})$ that is not obviously isomorphic to the standard toric ring $S[\mathbf{C}\sigma \cap (\Lambda_\sigma \oplus \mathbb{Z})]$. An isomorphism can however be easily defined by mapping z^m to $\tilde{\bar{s}}_{\sigma\tau}(m)z^m$ for τ the minimal cell with $m \in \mathbf{C}\tau$.

Construction of $\mathfrak{X}^\circ$. Assuming the wall structure $\mathscr{S}$ is consistent for the gluing data $\mathbf{s}$ in codimension zero and one the construction of $\mathfrak{X}^\circ$ in Proposition 2.16 is unchanged with the new definition of the localization homomorphism $\chi_{\mathfrak{b},\mathfrak{u}} = \chi_{\mathfrak{b},\mathfrak{u}}(\mathbf{s})$ in (5.2). Note that the change of $\chi_{\mathfrak{b},\mathfrak{u}}$ implicitly also changes the isomorphism $\theta_{\mathfrak{j}}$ for crossing a codimension one joint (2.22).

We then have the analogue of Proposition 2.19:

Proposition 5.15. *Suppose* $\mathrm{ob}_{\mathbb{P}}(\bar{\mathbf{s}}) = 0$ *and* $\tilde{\bar{\mathbf{s}}}$ *is a lift of* $\bar{\mathbf{s}}$*, yielding* X_0*. Then the reduction of* $\mathfrak{X}^\circ$ *modulo* I_0 *is canonically isomorphic to the complement of the union of codimension two strata in* X_0*. In particular,* $\mathfrak{X}^\circ$ *is separated as a scheme over* $A[Q]/I$*.*

Proof. Taking $I = I_0$, it is enough to construct suitable maps $\psi_{\mathfrak{b}} : \operatorname{Spec} R_{\mathfrak{b}} \to X_0$ and $\psi_{\mathfrak{u}} : \operatorname{Spec} R_{\mathfrak{u}} \to X_0$, for all slabs $\mathfrak{b}$ and chambers $\mathfrak{u}$, such that whenever $\mathfrak{b} \subseteq \mathfrak{u}$, we have a commutative diagram

$$\begin{array}{ccc}\operatorname{Spec} R_{\mathfrak{u}} & \xrightarrow{\ \psi_{\mathfrak{u}}\ } & X_0\\ {\scriptstyle \chi_{\mathfrak{b},\mathfrak{u}}}\downarrow & \nearrow{\scriptstyle \psi_{\mathfrak{b}}} & \\ \operatorname{Spec} R_{\mathfrak{b}} & & \end{array} \tag{5.11}$$

with $\psi_{\mathfrak{u}}$, $\psi_{\mathfrak{b}}$ being open immersions. To describe these maps, suppose $\mathfrak{b} \subseteq \underline{\rho} \subseteq \rho$, with $\rho \subseteq \sigma, \sigma' \in \mathscr{P}_{\max}$. Choose a point $v \in B(\frac{1}{d}\mathbb{Z}) \cap \operatorname{Int}(\mathbf{C}\rho)$ for some $d > 0$.

By using the affine chart defined in a neighbourhood of $\operatorname{Int}\underline{\rho}$ to embed $\sigma\cup\sigma'$ into an affine space, for any $d'>0$, any point $m\in B(\frac{1}{dd'}\mathbb{Z})\cap(\mathbf{C}\sigma\cup\mathbf{C}\sigma')$ yields a tangent vector $m-d'v\in\Lambda_x$, for $x\in\operatorname{Int}\underline{\rho}$. Now the image of $\psi_{\mathfrak{b}}$ will be the open affine subset $U_\rho:=\operatorname{Spec}(S[B](\tilde{\tilde{\mathbf{s}}}))_{(z^v)}$ of X_0, where the localization is in degree 0. The map $\psi_{\mathfrak{u}}$ can be defined as follows. The localized ring $(S[B](\tilde{\tilde{\mathbf{s}}}))_{(z^v)}$ is generated as an $A[Q]/I_0$-module by elements $z^m/(z^v)^{d'}$ for m ranging over elements of $B(\frac{1}{dd'}\mathbb{Z})\cap\operatorname{Int}(\mathbf{C}\sigma\cup\mathbf{C}\sigma')$ for any $d'>0$. For any such m, one can write $m-d'v=m'+a\xi$ where $m'\in\Lambda_\rho$, $a\in\mathbb{Z}$ and $\xi=\xi(\rho)\in\Lambda_x$ is the chosen tangent vector pointing into σ. Then

$$\psi_{\mathfrak{b}}^*(z^m/(z^v)^{d'})=\begin{cases} z^{m'} & a=0\\ s_{\sigma\underline{\rho}}(m'+a\xi)^{-1}z^{m'}Z_+^a & a>0\\ s_{\sigma'\underline{\rho}}(m'+a\xi)^{-1}z^{m'}Z_-^{-a} & a<0.\end{cases}$$

Similarly, if $\mathfrak{u}\subseteq\sigma$, we define $\psi_{\mathfrak{u}}$ by

$$\psi_{\mathfrak{u}}^*(z^m/(z^v)^{d'})=\begin{cases} s_{\sigma\underline{\rho}}(m')z^{m'} & a=0\\ z^{m'+a\xi} & a>0\\ 0 & a<0,\end{cases}$$

with a similar definition reversing the roles of $a>0$ and $a<0$ if $\mathfrak{u}\subseteq\sigma'$. One checks easily that these two maps are ring homomorphisms and that (5.11) commutes. □

Modification of broken lines. If sums over broken lines are to extend the definition of (certain) monomials z^m in the construction of X_0 to $\mathfrak{X}^\circ$ they need to be modified by closed gluing data analogously. Recall from Remark 3.4,1 that each broken line β defines an asymptotic monomial m. Denote by $\mathscr{P}_m\subseteq\mathscr{P}$ the polyhedral subcomplex consisting of all cells τ and their faces having m as an asymptotic monomial. Each maximal cell in $\mathscr{P}_m$ could be the starting cell of a broken line contributing to ϑ_m. Passing between neighbouring maximal cells $\sigma,\sigma'\in\mathscr{P}_m$ through $\underline{\rho}\subseteq\sigma\cap\sigma'$, the initial coefficient has to change by $s_{\sigma\underline{\rho}}^{-1}(\overline{m})\cdot s_{\sigma'\underline{\rho}}(\overline{m})\in A^\times$. The evaluations $s_{\sigma\underline{\rho}}(\overline{m})$ for $\sigma,\underline{\rho}\subseteq|\mathscr{P}_m|$ define a one-cocycle on $\mathscr{P}_m$ with values in $A^\times$, whose class in $H^1(\mathscr{P}_m,A^\times)$ is an obstruction for a consistent choice of starting data of broken lines with asymptotic monomial m. Once this obstruction vanishes the choice of a maximal cell $\sigma\in\mathscr{P}_m$ gives finitely many distinguished normalizations of ϑ_m by only requiring the starting coefficient a_1 for broken lines asymptotically contained in σ to be 1. The starting coefficient on a different maximal cell $\sigma'\in\mathscr{P}_m$ is then uniquely determined from the open gluing data by consistency. Thus in this case, the definition of normalized in Definition 3.3 is replaced by a_1 being a product of $s_{\sigma''\underline{\rho}}(\overline{m})$ and their inverses for $\sigma'', \underline{\rho} \subseteq |\mathscr{P}_m|$.

Fortunately, $\mathscr{P}_m$ is usually contractible and hence the obstruction vanishes. This is for example the case under the natural assumption that B is asymptotically convex in the sense that for each asymptotic monomial m there is a unique minimal cell τ carrying it. Thus τ is contained in any other cell on which m is an asymptotic monomial and hence $\mathscr{P}_m$ retracts to τ. For example, convexity is trivially true in the two-dimensional conical case of [GHK1]. Contractibility of $\mathscr{P}_m$ also holds for asymptotic monomials of degree $d > 0$ on $\mathbf{C}B$ because in this case $\mathscr{P}_m$ retracts to an open set of the form W_τ of B, and such a set is contractible.

With this modification all arguments in Section 3 go through.

Remark 5.16. An alternative view on twisting the construction via gluing data runs as follows. Recall that $\mathcal{P}$ defines an extension of Λ by the constant sheaf $\underline{Q}^{\mathrm{gp}}$. Now $\mathcal{E}xt^1(\Lambda, \underline{A}^\times) = 0$ since Λ is locally free, and hence $H^1(B_0, \check{\Lambda} \otimes_{\mathbb{Z}} \underline{A}^\times) = \mathrm{Ext}^1_{B_0}(\Lambda, \underline{A}^\times)$ by the local to global spectral sequence. Therefore an equivalence class of open gluing data $\mathbf{s}$ yields an equivalence class of extensions

$$0 \longrightarrow \underline{A}^\times \longrightarrow \mathcal{P}' \longrightarrow \Lambda \longrightarrow 0$$

of abelian sheaves on B_0. Then $\mathcal{P}_{\mathbf{s}} := \mathcal{P} \times_\Lambda \mathcal{P}'$ is an extension of Λ by $\underline{Q}^{\mathrm{gp}} \oplus \underline{A}^\times$:

$$0 \longrightarrow \underline{Q}^{\mathrm{gp}} \oplus \underline{A}^\times \longrightarrow \mathcal{P}_{\mathbf{s}} \longrightarrow \Lambda \longrightarrow 0. \tag{5.12}$$

Since Λ is locally free this sequence splits locally and we have (non-canonical) isomorphisms $\mathcal{P}_{\mathbf{s},x} \simeq Q^{\mathrm{gp}} \oplus A^\times \oplus \Lambda_x$ for $x \in B_0$. Define $\mathcal{P}^+_{\mathbf{s}} \subseteq \mathcal{P}_{\mathbf{s}}$ as the preimage of $\mathcal{P}^+ \to \mathcal{P}$ under the projection $\mathcal{P}_{\mathbf{s}} \to \mathcal{P}$. From $\mathcal{P}^+_{\mathbf{s}}$ we can first define a locally constant sheaf of rings $\mathcal{R}$ with fibres isomorphic to $(A[Q]/I)[\Lambda_x]$ as follows. Denote by $A[\mathcal{P}^+_{\mathbf{s}}]$ the sheaf of monoid rings with exponents in the stalks $\mathcal{P}^+_{\mathbf{s},x}$ and coefficients in A. Clearly, $A[\mathcal{P}^+_{\mathbf{s}}]$ is a sheaf of $A[Q]$-algebras, and hence $I \subseteq A[Q]$ defines a sheaf of ideals $\mathcal{I} \subseteq A[\mathcal{P}^+_{\mathbf{s}}]$. Moreover, there is an embedding of $\underline{A}^\times$ into $A[\mathcal{P}^+_{\mathbf{s}}]$ by mapping $a \in A^\times$ to $a^{-1} \cdot z^{(0,a)}$ with $(0, a)$ viewed as a section of $\mathcal{P}^+_{\mathbf{s}}$ via (5.12). The induced action leaves $\mathcal{I}$ invariant and hence descends to the quotient. We may therefore define

$$\mathcal{R} := \big(A[\mathcal{P}^+_{\mathbf{s}}]/\mathcal{I}\big)/\underline{A}^\times.$$

Note that by the local description of $\mathcal{P}^+_{\mathbf{s}}$ this sheaf of rings has the predicted stalks. One can then define our rings for maximal cells by $R_\sigma := \Gamma(\mathrm{Int}\,\sigma, \mathcal{R})$. For codimension one the analogue of $(A[Q]/I)[\Lambda_\rho]$ in (2.17), which hosts $f_{\mathfrak{b}}$, is the $A[Q]$-subalgebra R_ρ of $\mathcal{R}_x$ generated by Λ_ρ, for some $x \in \mathrm{Int}\,\underline{\rho}$. For $Z_\pm \in \mathcal{R}_x$ take some lifts of complementary vectors $\pm\xi \in \Lambda_x$ to $(\mathcal{P}_{\mathbf{s}})_x$ with $Z_+Z_- = z^{\kappa_{\underline{\rho}}}$ in

$\mathcal{R}_x$. Now $R_\mathfrak{b}$ can be defined in analogy with (2.17) by

$$R_\mathfrak{b} = R_\rho[Z_+, Z_-]/\big(Z_+Z_- - f_\mathfrak{b} z^{\kappa_{\underline{\rho}}}\big).$$

Note that while $Z_\pm$ depends on choices, $R_\mathfrak{b}$ is defined invariantly as a subring of a localization of $\mathcal{R}_y$ for y close to $x \in \operatorname{Int} \underline{\rho}$ in a maximal cell. From this point of view the localization morphism $\chi_{\mathfrak{b},\mathfrak{u}}$ is again defined canonically, this time by parallel transport inside the locally constant sheaf $\mathcal{R}$. The twist in comparison with trivial gluing data comes from global non-triviality of the extension (5.12).

The next topic concerns consistency of the lifting of a wall structure from B to $\mathbf{C}B$, generalizing Proposition 4.9.

Proposition 5.17. *Let $\mathscr{S}$ be a wall structure on $(B,\mathscr{P})$ that is consistent for gluing data $\mathbf{s}$ on B, with induced closed gluing data $\bar{\mathbf{s}}$. If $\tilde{\mathbf{s}}$ and $\tilde{\bar{\mathbf{s}}}$ are as given by Proposition 5.12, then the lifted wall structure $\mathbf{C}\mathscr{S}$ on $(\mathbf{C}B, \mathbf{C}\mathscr{P})$ is consistent as well.*

Proof. We reexamine the proof of Proposition 4.9. For consistency in codimension zero the gluing data play no role and the proof works as before. In codimension one we distinguished two kinds of monomials, those lifted from B (of degree zero) and one of the form $z^{(m,a)}$ with $a > 0$ and with (m,a) tangent to the joint. Consistency for monomials of degree zero follows as before by observing that $\tilde{\mathbf{s}}$ restricts to $\mathbf{s}$ on Λ.

For $z^{(m,a)}$ equation (4.5) still holds, while (4.6) now reads

$$(\theta \circ \chi_{\mathfrak{b}_1,\sigma}, \theta' \circ \chi_{\mathfrak{b}_1,\sigma'})(z^m) = (\chi_{\mathfrak{b}_2,\sigma}, \chi_{\mathfrak{b}_2,\sigma'})(h),$$

still with $h = f \cdot z^m$ and $f \in 1 + I_0 \cdot R_{\mathfrak{b}_2}$. The rest of the argument remains unchanged.

For the argument in codimension two consistency of $z^{(0,1)}$ follows from Proposition 5.8 characterizing the vanishing of the obstruction class $\mathrm{ob}_\Delta(\tilde{\mathbf{s}})$ in terms of consistency for monomials tangent to codimension two cells. Note that the associated local regular function $\vartheta^{\mathfrak{j}}_{(0,1)}(p)$ for p in a chamber $\mathfrak{u}$ now restricts to $a_\mathfrak{u} z^{(0,1)}$ in $R_\mathfrak{u}$ for some $a_\mathfrak{u} \in A^\times$. Non-trivial closed gluing data around $\mathfrak{j}$ are reflected by $a_\mathfrak{u} \neq 1$. □

The main result Theorem 4.12 now generalizes. Suppose given $\mathbf{s}$ projective consistent open gluing data, $\tilde{\bar{\mathbf{s}}}$ a choice of lift of the induced closed gluing data $\bar{\mathbf{s}}$ with $\tilde{\mathbf{s}}$ the corresponding lift of open gluing data to $\mathbf{C}B$. If $\mathscr{S}$ is a consistent wall structure for $\mathbf{s}$, we obtain schemes $\mathfrak{X}^\circ$, $\mathfrak{Y}^\circ$ from the structures $\mathscr{S}$ and $\mathbf{C}\mathscr{S}$ respectively. This gives rise to rings R_∞ and S as in Theorem 4.12, and $\mathfrak{W} := \operatorname{Spec} R_\infty$, $\mathfrak{Y} := \operatorname{Spec} S$, and $\mathfrak{X} := \operatorname{Proj} S$. These satisfy all the same properties that the corresponding schemes in the statement of Theorem 4.12 satisfy. We note, however, that the line bundle $\mathfrak{L} \to \mathfrak{X}$ depends on the choice of lift $\tilde{\bar{\mathbf{s}}}$ of $\bar{\mathbf{s}}$, and it is not difficult to see two choices of lift $\tilde{\bar{\mathbf{s}}}$, $\tilde{\bar{\mathbf{s}}}'$ define isomorphic line bundles over $\mathfrak{X}$ if and only if the two lifts are equivalent in the sense of Definition 5.11.

Remark 5.18. It is worthwhile emphasizing that the projectivity of the construction only depends on projectivity of the central fibre X_0 over W_0, where in the notation of Theorem 4.12 the affine scheme $W_0 \subseteq \mathfrak{W}$ is the fibre over $\operatorname{Spec}\big(A[Q]/I_0\big) \subseteq \operatorname{Spec}(A[Q]/I)$. Projectivity then automatically continues to hold for $\mathfrak{X} \to \mathfrak{W}$.

In concluding this section let us emphasize that with trivial gluing data, X_0 is the pull-back of a scheme over $\operatorname{Spec}\mathbb{Z}$ to $\operatorname{Spec} A$. Without gluing data it is therefore impossible to produce non-trivial, locally trivial deformations. Only if all locally trivial deformations are trivial can one hope to retrieve all deformations already with trivial gluing data. This is the case for example in the projective smoothing of X_0 with all irreducible components $\mathbb{P}^2$, see [GHKS].

6. Abelian varieties and other examples

We will discuss several extended examples. The longest is a discussion of abelian varieties. The main point is to compare our construction with classical theta functions: this motivates the use of the term "theta function." In particular, we will show that in this case our theta functions equal the classical theta functions up to some explicit rescaling. We then look at some examples with very complex wall structures, but for which we can nevertheless say something non-trivial.

Example 6.1. Continuing with Example 1.13, taking $B = M_{\mathbb{R}}/\Gamma$, P, Q and φ_0 as given there, let $I_0 = Q\setminus\{0\}$. In this case the empty wall structure is consistent, so we obtain for any monomial ideal I of Q with radical I_0 a projective family $\mathfrak{X} \to \operatorname{Spec}\Bbbk[Q]/I$ with $\mathfrak{X} = \operatorname{Proj} S$ and with theta functions $\vartheta_m \in \Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}(d))$ for $m \in B(\frac{1}{d}\mathbb{Z})$. Taking the inverse limit of the rings S over all ideals with radical I_0 gives a graded ring $\widehat{S}$ over the completion $\widehat{\Bbbk[Q]}$ of $\Bbbk[Q]$ with respect to the ideal I_0, and hence a projective family $\mathcal{X} = \operatorname{Spec}\widehat{S} \to \operatorname{Spec}\widehat{\Bbbk[Q]}$. This is of course a degenerating family of abelian varieties, a variant of constructions of Mumford [Mu2] and Alexeev [Al].

Before examining this in more detail, let us first obtain a better understanding of the function φ_0. The periodicity relation (1.8) generalizes to a periodicity relation for φ_0 given by

$$\varphi_0(x+\gamma) = \varphi_0(x) + \alpha_\gamma(x) \tag{6.1}$$

for $\alpha_\gamma : M \to Q^{\mathrm{gp}}$ an affine linear function. We can write α_γ as a sum of a linear function and a constant, $\alpha_\gamma = d\alpha_\gamma + c_\gamma$. Applying (6.1) for $\gamma = \gamma_1, \gamma_2$ and $\gamma_1 + \gamma_2$ gives

$$\alpha_{\gamma_1+\gamma_2} = d\alpha_{\gamma_1} + d\alpha_{\gamma_2} + (d\alpha_{\gamma_1}(\gamma_2) + c_{\gamma_1} + c_{\gamma_2}),$$

so in particular $\overline{Z} : \Gamma \times \Gamma \to Q^{\mathrm{gp}}$ given by $\overline{Z}(\gamma_1, \gamma_2) = d\alpha_{\gamma_1}(\gamma_2)$ is a symmetric form. This gives rise to a quadratic function

$$\bar{\varphi}_0 : M_{\mathbb{R}} \to Q^{\mathrm{gp}}_{\mathbb{R}}, \quad \bar{\varphi}_0(x) = \frac{1}{2}\overline{Z}(x, x),$$

satisfying the periodicity condition

$$\bar{\varphi}_0(x + \gamma) = \bar{\varphi}_0(x) + d\alpha_\gamma(x) + \frac{1}{2}\overline{Z}(\gamma, \gamma).$$

Choose a basis $e_1, \ldots, e_n$ of M such that Γ is generated by $\{f_i = d_i e_i\}$, $d_1 | \cdots | d_n$ positive integers. Denote by $f_1^*, \ldots, f_n^*$ the dual basis. As φ_0 can be changed by an affine linear function without affecting the construction, we can replace φ_0 with

$$\varphi_0 + \sum_{i=1}^{n} (\frac{1}{2}\overline{Z}(f_i, f_i) - c_{f_i}) f_i^* - \varphi_0(0). \tag{6.2}$$

This has the effect[14] of replacing c_γ with $\frac{1}{2}\overline{Z}(\gamma, \gamma)$ as the constant part of α_γ. As a consequence, $\bar{\varphi}_0 - \varphi_0$ is a single-valued function ψ on B. Note that φ_0 may no longer have integral slopes (that is, slopes in $N \otimes Q^{\mathrm{gp}}$) but after rescaling φ_0 (which has the effect of base-changing the construction), we may assume it continues to have integral slope. This allows us to assume a standard form for the c_γ.

In any event, regardless of the choice of φ_0, there is a standard description of the family $\widehat{\mathfrak{X}} \to \operatorname{Spf} \widehat{\Bbbk[Q]}$ as the quotient of a (non-finite type) fan, as follows. Consider in $M_{\mathbb{R}} \times Q^{\mathrm{gp}}_{\mathbb{R}}$ the polytope

$$\Xi_{\varphi_0} := \{(m, \varphi_0(m) + q) \mid q \in Q_{\mathbb{R}}\},$$

where $Q_{\mathbb{R}} = \operatorname{Hom}(P, \mathbb{R}_{\geq 0})$ and $Q = Q_{\mathbb{R}} \cap Q^{\mathrm{gp}}$. There is a lift of the Γ-action on $M_{\mathbb{R}}$ to $M_{\mathbb{R}} \times Q^{\mathrm{gp}}_{\mathbb{R}}$ leaving Ξ_{φ_0} invariant by letting $\gamma \in \Gamma$ act

$$(m, q) \longmapsto (m + \gamma, q + \alpha_\gamma(m)).$$

Let Σ be the normal fan to Ξ_{φ_0} in $N_{\mathbb{R}} \times P^{\mathrm{gp}}_{\mathbb{R}}$ (with $N = \operatorname{Hom}(M, \mathbb{Z})$). The one-dimensional rays of Σ are dual to maximal faces of Ξ_{φ_0}. If $\sigma \in \bar{\mathscr{P}}_{\max}$ and τ is a codimension one face of $Q_{\mathbb{R}}$, then

$$\{(m, \varphi_0(m) + q) \mid m \in \sigma, q \in \tau\}$$

[14] This claim comes down to showing that $\frac{1}{2}\bar{Z}(\gamma, \gamma) = c_\gamma + \sum_{i=1}^n (\frac{1}{2}\bar{Z}(f_i, f_i) - c_{f_i}) f_i^*(\gamma)$. This can be proved by induction, showing that the equation holds for $\gamma \pm f_i$ if it holds for γ. The computation is straightforward, but is omitted for its length.

is a maximal face of Ξ_{φ_0} and all maximal faces are of this form. The primitive normal vector to this face is $(-d(\varphi_0|_\sigma)^t(p_\tau), p_\tau)$, where $d(\varphi_0|_\sigma)$ is viewed as an element of $\mathrm{Hom}(M, Q^{\mathrm{gp}})$ and its transpose as an element of $\mathrm{Hom}(P^{\mathrm{gp}}, N)$, and $p_\tau \in P$ is the primitive generator of the edge of $P_{\mathbb{R}} = \mathrm{Hom}(Q, \mathbb{R}_{\geq 0})$ corresponding to the face τ of $Q_{\mathbb{R}}$.

In particular, the projection $N_{\mathbb{R}} \times P_{\mathbb{R}}^{\mathrm{gp}} \to P_{\mathbb{R}}^{\mathrm{gp}}$ defines a map of fans from Σ to the fan of faces of $P_{\mathbb{R}}$. If X_Σ denotes the toric variety (not of finite type) defined by Σ, we obtain a flat morphism $f : X_\Sigma \to \operatorname{Spec} \Bbbk[Q]$. The action of Γ on Ξ_{φ_0} induces an action on Σ by taking the transpose of its linear part. Thus $\gamma \in \Gamma$ acts by $(n, p) \mapsto (n + (d\alpha_\gamma)^t(p), p)$. Here we view $d\alpha_\gamma$ as a homomorphism $M \to Q^{\mathrm{gp}}$, and its transpose $(d\alpha_\gamma)^t$ accordingly as a homomorphism $P^{\mathrm{gp}} \to N$. In turn, Γ acts on $X_\Sigma \times_{\operatorname{Spec} \Bbbk[Q]} \operatorname{Spec} \Bbbk[Q]/I$ for any monomial ideal I with radical I_0, and the quotient

$$(X_\Sigma \times_{\operatorname{Spec} \Bbbk[Q]} \operatorname{Spec} \Bbbk[Q]/I)/\Gamma \to \operatorname{Spec} \Bbbk[Q]/I \tag{6.3}$$

can be seen to coincide with $\mathfrak{X} \to \operatorname{Spec} \Bbbk[Q]/I$.

Using this description, theta functions on $\mathfrak{X}$ are traditionally seen by extending the action of Γ on $M \times Q^{\mathrm{gp}}$ to an action of Γ on $M \times Q^{\mathrm{gp}} \times \mathbb{Z}$ which preserves the cone $C(\Xi_{\varphi_0})$ and the last factor $\mathbb{Z}$. This lifts the Γ-action on X_Σ to the total space of the line bundle $\tilde{\mathcal{L}}$ induced by the polytope Ξ_{φ_0}. This lifting is given as follows. The action on $M \times Q^{\mathrm{gp}}$ is given by γ acting via $(m, q) \mapsto (m, q + d\alpha_\gamma(m) + c_\gamma)$ with $c_\gamma \in Q^{\mathrm{gp}}$ the constant part of α_γ as before. The extension is then given by

$$T_\gamma : (m, q, r) \mapsto (m + r\gamma, q + d\alpha_\gamma(m) + rc_\gamma, r).$$

Now for $r \in \mathbb{Z}_{>0}$, the integral points of $r\Xi_{\varphi_0}$ correspond to sections of the line bundle $\tilde{\mathcal{L}}^{\otimes r}$ on X_Σ. Using the action of Γ on the total space of $\tilde{\mathcal{L}}$, compatible with the action of Γ on X_Σ, the line bundle $\tilde{\mathcal{L}}$ descends to a line bundle $\mathcal{L}$ on $\mathfrak{X}$. Similarly, Γ-invariant sections of $\tilde{\mathcal{L}}$ descend to sections of $\mathcal{L}$. We then obtain theta functions on $\mathfrak{X} \to \operatorname{Spec} \Bbbk[Q]/I$ by constructing Γ-invariant sections on $X_\Sigma \times_{\operatorname{Spec} \Bbbk[Q]} \operatorname{Spec} \Bbbk[Q]/I$: for $m \in B(\frac{1}{r}\mathbb{Z})$, we have

$$\vartheta_m = \sum_{\gamma \in \Gamma} z^{T_\gamma(rm, r\varphi_0(m), r)} = \sum_{\gamma \in \Gamma} z^{(r(m+\gamma), r\varphi_0(m+\gamma), r)}. \tag{6.4}$$

Before comparing this formula with the one given by jagged paths, let us compare the above formula with the classical notion of theta function; indeed, it is this comparison which justifies the use of the term “theta function.” To do so, we work complex analytically, with $\Bbbk = \mathbb{C}$. There is some analytic open neighbourhood $S \subseteq \operatorname{Spec} \Bbbk[Q]$ of the zero-dimensional stratum such that the action of Γ on $f^{-1}(S)$ is free, giving $\bar{f} : \mathcal{X} := f^{-1}(S)/\Gamma \to S$ an analytic degeneration of abelian varieties. For $p \in S \cap \operatorname{Spec} \Bbbk[Q^{\mathrm{gp}}] \subseteq P^{\mathrm{gp}} \otimes \mathbb{G}_m$, $f^{-1}(p)$ is canonically the algebraic torus $N\otimes\mathbb{G}_m$ and $\bar f^{-1}(p)$ is the abelian variety $(N\otimes\mathbb{G}_m)/\Gamma$ where $\Gamma\hookrightarrow N\otimes\mathbb{G}_m$ via $\gamma\mapsto(d\alpha_\gamma)^t(p)$.

In terms of a period matrix, note the choice of basis $\{f_i\}$ define coordinates $u_1,\ldots,u_n$ on $N\otimes\mathbb{C}$ via pairing with f_i. Said differently, u_i is the pull-back of $z^{f_i}\in\mathbb{C}[M]$ via the exponential map

$$N\otimes\mathbb{C}\longrightarrow N\otimes\mathbb{G}_m,\quad \sum_i e_i^*\otimes\lambda_i\longmapsto\sum_i e_i^*\otimes e^{2\pi\sqrt{-1}\lambda_i}.$$

Note that the kernel of this map is generated by $e_1^*=d_1f_1^*,\ldots,e_n^*=d_nf_n^*$. Then using the coordinates $\{u_i\}$, one sees that $\bar f^{-1}(p)$ has period matrix $(D,Z(p))$, where $D=\mathrm{Diag}(d_1,\ldots,d_n)$ and

$$Z(p)_{ij}=\langle(d\alpha_{f_j})(f_i),\frac{\log p}{2\pi\sqrt{-1}}\rangle.$$

Here log is a local choice of inverse to $\exp: P^{\mathrm{gp}}\otimes\mathbb{C}\to P^{\mathrm{gp}}\otimes\mathbb{G}_m$. This is the matrix in the basis $\{f_i^*\}$ for the $\mathbb{C}$-valued bilinear form $Z(p)(\gamma_1,\gamma_2)=\langle\bar Z(\gamma_1,\gamma_2),\frac{\log p}{2\pi\sqrt{-1}}\rangle$. Note that $Z(p)$ defines the period point in the Siegel upper half space $\mathfrak{H}_n=\{Z\in M(n,\mathbb{C})\,|\,Z=Z^t,\ \mathrm{Im}\,Z>0\}$.

In what follows, we will assume that $c_\gamma=\frac{1}{2}\bar Z(\gamma,\gamma)$ and $\varphi_0(0)=0$, as is achieved by (6.2). Now the line bundle $\tilde{\mathcal{L}}$ on X_Σ is trivialized when restricted to $f^{-1}(p)$, and we can choose, say, $z^{(0,\varphi_0(0),1)}=z^{(0,0,1)}$ as a trivializing section. Then as a regular function on $f^{-1}(p)$, for $m\in B(\mathbb{Z})$, the theta function ϑ_m takes the form $\sum_{\gamma\in\Gamma}z^{m+\gamma}z^{\varphi_0(m+\gamma)}(p)$. Writing this as a function on $N\otimes\mathbb{C}$ and using $\zeta=\sum_i u_if_i^*$, gives an expression as a function of the u_i

$$\vartheta_m=\sum_{\gamma\in\Gamma}z^{\varphi_0(m+\gamma)}(p)\exp(2\pi\sqrt{-1}\langle\zeta,m+\gamma\rangle).$$

Finally, writing $\varphi_0(m+\gamma)=\psi(m)+\bar\varphi_0(m+\gamma)$ (as ψ is single-valued), this becomes

$$\vartheta_m=z^{\psi(m)}(p)\sum_{\gamma\in\Gamma}\exp(\pi\sqrt{-1}Z(p)(m+\gamma,m+\gamma)+2\pi\sqrt{-1}\langle\zeta,m+\gamma\rangle).$$

Except for the scale factor $z^{\psi(m)}(p)$, after writing the exponent in terms of the basis f_i, this gives the standard form for the classical theta function $\vartheta\begin{bmatrix}c^1\\0\end{bmatrix}(\zeta,Z)$ where $c^1=(m_1,\ldots,m_n)$, $m=\sum m_if_i$, see e.g. [BiLa], p.223).

We now compare the formula (6.4) for ϑ_m with the description given by jagged paths. Fix any maximal cell $\sigma\in\mathscr{P}$ and $p\in\sigma$ a chosen basepoint. For $m\in B(\frac{1}{r}\mathbb{Z})$, the set of all jagged paths from m to p is easily described via a factorization through the universal cover $\pi:M_\mathbb{R}\to M_\mathbb{R}/\Gamma$ of B. Fixing one lift $\tilde m$ of m to $M_\mathbb{R}$, any lift $\tilde p$ of p to $M_\mathbb{R}$ yields a jagged path $\tilde\gamma$ whose image is just a straight line joining $\tilde m$ to $\tilde p$. The composition with π gives a jagged path in B.

The resulting theta function, described as a sum of monomials indexed by jagged paths, is easily compared with (6.4). Indeed, the sheaf Λ on B is just the constant sheaf with stalk M, the sheaf $\mathcal{P}$ has stalk $M \times Q^{\rm gp}$ with monodromy the linear part of the action of Γ on $M\times Q^{\rm gp}$ by affine transformations described above, and the sheaf $\widetilde{\mathcal{P}}$ has stalk $M\times Q^{\rm gp}\times \mathbb{Z}$ with monodromy given by the action of Γ on this latter group as described above. Thus, after choosing a local representative for φ_0 near m, say the representative given by φ_0 in a neighbourhood of $\tilde{m}$, we see $(r\tilde{m}, r\varphi_0(\tilde{m}), r)$ represents $r\cdot\varphi_{0*}(\mathrm{ev}_m)$, and $T_\gamma(r\tilde{m}, r\varphi_0(\tilde{m}), r)$ represents parallel transport of $r\cdot\varphi_{0*}(\mathrm{ev}_m)$ around a loop corresponding to $\gamma\in\Gamma$. In particular, after parallel transport to p, we see that ϑ_m coincides with ϑ_m as defined using jagged paths, see §4.5.

The theta functions constructed above are not all theta functions on an abelian variety. Indeed, given an ample line bundle $\mathcal{L}$ on an abelian variety A, and $t_x : A \to A$ denoting translation by an element $x\in A$, $t_x^*\mathcal{L}$ is isomorphic to $\mathcal{L}$ only for finitely many values of x. To see this write $K(\mathcal{L})$ for the kernel of the map $A\to A^\vee$ given by $x\mapsto (t_x^*\mathcal{L})\otimes\mathcal{L}^{-1}$. By ampleness of $\mathcal{L}$ this kernel is a finite group.

To identify such line bundles in our construction, we need to use gluing data as in §5. Indeed, we continue to use trivial gluing data to construct our family $\mathcal{X}\to \operatorname{Spec}\widehat{\Bbbk[Q]}$, but there is a choice of lifting trivial gluing data to $\mathbf{C}B$ as described in Proposition 5.12. This tells us that the set of liftings $\tilde{\mathbf{s}}$ of the trivial gluing data $\mathbf{s}$ up to equivalence is canonically in bijection with $H^1(B,\Bbbk^\times)$.

The technically easiest way to think about such a choice of lifts is to use the description of $\widehat{\mathfrak{X}}\to \operatorname{Spf}\widehat{\Bbbk[Q]}$ as a quotient, and to any finite order, we can use (6.3). In particular, the cohomology group $H^1(B,\Bbbk^\times)$ can be represented by the group cohomology $H^1(\Gamma,\Bbbk^\times)$, with trivial action of Γ on $\Bbbk^\times$. Then $H^1(\Gamma,\Bbbk^\times)\cong \operatorname{Hom}(\Gamma,\Bbbk^\times)$, so we view gluing data $\tilde{\mathbf{s}}$ lifting the trivial open gluing data as a map $\tilde{\mathbf{s}}:\Gamma\to\Bbbk^\times$. We can then use this to twist the action of Γ on the line bundle $\tilde{\mathcal{L}}$ on X_Σ, by γ acting on a monomial section $z^{(m,q,r)}$ of $\tilde{\mathcal{L}}^{\otimes r}$ by taking it to $\tilde{\mathbf{s}}(\gamma)z^{T_\gamma(m,q,r)}$. Thus again we can look at Γ-invariant sections under this new action, getting for $m\in B(\frac{1}{r}\mathbb{Z})$,

$$\vartheta_m=\sum_{\gamma\in\Gamma}\tilde{\mathbf{s}}(\gamma)z^{(r(m+\gamma),r\varphi_0(m+\gamma),r)}.$$

Again, before comparing this with what we get from broken lines, let us compare this expression with classical theta functions. Restricting to a point $p\in S\cap \operatorname{Spec}\Bbbk[Q^{\rm gp}]$ as before, this becomes

$$\vartheta_m = z^{\psi(m)}(p)\sum_{\gamma\in\Gamma}\tilde{\mathbf{s}}(\gamma)\exp(\pi\sqrt{-1}Z(p)(m+\gamma,m+\gamma)+2\pi\sqrt{-1}\langle\zeta,m+\gamma\rangle). \tag{6.5}$$

This can be interpreted as a classical theta function as follows. Because the period matrix of $f^{-1}(p)$ is $(D, Z(p))$, viewing $Z(p)$ as giving a map $Z(p) : M_{\mathbb{R}} \to N_{\mathbb{C}}$ via $Z(p)(m) = Z(p)(m, \cdot)$, any element of $N_{\mathbb{C}}$ can be written uniquely as $Z(p)c^1 + c^2$ for some vectors $c^1 \in M_{\mathbb{R}}$, $c^2 \in N_{\mathbb{R}}$. In particular, there are such vectors c^1, c^2 such that $\tilde{\mathbf{s}}(\gamma) = \exp(2\pi\sqrt{-1}\langle Z(p)c^1 + c^2, \gamma\rangle)$ for all $\gamma \in \Gamma$. Thus we can write

$$\vartheta_m = z^{\psi(m)}(p) \sum_{\gamma\in\Gamma} \frac{\exp\left(\pi\sqrt{-1}Z(p)(m+\gamma+c^1, m+\gamma+c^1) + 2\pi\sqrt{-1}\langle\zeta + c^2, m+\gamma+c^1\rangle\right)}{\exp\left(\pi\sqrt{-1}Z(p)(c^1,c^1) + 2\pi\sqrt{-1}(Z(p)(m,c^1) + \langle\zeta+c^2,c^1\rangle + \langle c^2, m\rangle)\right)}.$$

We notice the denominator is nowhere zero and independent of γ. Recall ϑ_m defines a section of a line bundle on $f^{-1}(p)$ by trivializing the pull-back of the line bundle to the universal cover $N \otimes \mathbb{C}$. A different choice of trivialization is determined by an entire invertible function. In particular, after changing this trivialization (thereby changing the factor of automorphy determining the line bundle, see [BiLa], §2.1), the above function describes the same section of a line bundle as

$$\begin{aligned} &z^{\psi(m)}(p)\vartheta \begin{bmatrix} m + c^1 \\ c^2 \end{bmatrix} \\ =&z^{\psi(m)}(p) \sum_{\gamma\in\Gamma} \exp\left(\pi\sqrt{-1}Z(p)(m + \gamma + c^1, m + \gamma + c^1) + 2\pi\sqrt{-1}\langle\zeta + c^2, m + \gamma + c^1\rangle\right). \end{aligned}$$

To see that the expression (6.5) agrees with that given by jagged paths, it is easiest to use the description of parallel transport of monomials given by Remark 5.16. Indeed, the element $\tilde{\mathbf{s}} : \Gamma \to \Bbbk^\times$ defines an extension $\mathcal{P}'$ of Λ by $\underline{\Bbbk}^\times$. This extension is trivial when $\mathcal{P}'$ is pulled back to $M_{\mathbb{R}}$, with $\gamma \in \Gamma$ acting on $\Bbbk^\times \times M$ by $(s, m) \mapsto (s\tilde{\mathbf{s}}(\gamma), m+\gamma)$. Then parallel transport of monomials along jagged paths as already described in the case of trivial gluing data will provide the formula for ϑ_m in (6.5).

We note that this description of these more general theta functions is not really canonical either from the point of view of classical theta functions (as we need to change the trivialization on $N \otimes \mathbb{C}$, implying a change in factor of automorphy) or from the point of view of homological mirror symmetry. From the latter point of view, $B(\frac{1}{d}\mathbb{Z})$ is not the natural parameterizing set for theta functions, but rather this set translated by the vector c^1. Indeed, the Lagrangian mirror to $t_x^*\mathcal{L}$ should be a translate of the Lagrangian mirror to $\mathcal{L}$. The expectation from [PZ] is that the image under the SYZ fibration of the intersection points between this translated Lagrangian and the zero section of the SYZ fibration is this translated set. It is possible there is a more natural way to represent these theta functions corresponding to translated line bundles than done here.

Example 6.2. (Cf. [CPS], Example 2.4) Let B be the triangle in $\mathbb{R}^2$ with vertices $v_1 = (-1,-1)$, $v_2 = (-1,2)$ and $v_3 = (2,-1)$. Let $\mathscr{P}$ be the star decomposition of B, that is, each two-dimensional cell of $\mathscr{P}$ is the convex hull of 0 and an edge of B.

We will take the base monoid to be $Q = \mathbb{N}$, writing $\Bbbk[Q] = \Bbbk[t]$, and the PL function φ to be single-valued with

$$\varphi(0) = 0, \quad \varphi(-1,-1) = \varphi(-1,2) = \varphi(2,-1) = 1.$$

Note the sheaf $\mathcal{P}$ is the constant sheaf with coefficients $\mathbb{Z}^2 \oplus Q^{\mathrm{gp}}$, and we write the monomials $x := z^{(1,0,0)}$, $y := z^{(0,1,0)}$.

We construct a structure $\mathscr{S}$ for this data as follows. First consider the structure

$$\mathscr{S}_{\mathrm{in}} := \{(\rho_1, 1 + tx^{-1}y^{-1}), (\rho_2, 1 + tx^{-1}y^2), (\rho_3, 1 + tx^2y^{-1})\},$$

where ρ_i is the edge of $\mathscr{P}$ connecting 0 to v_i. Note that we do not bother to subdivide the ρ_i (hence abandoning the notation $\underline{\rho}_i$) as the affine structure does not have any singularities. By applying the Kontsevich-Soibelman lemma (see e.g., [Gr3], Theorem 6.38 for the simplest statement that incorporates the case needed here) we obtain a structure $\mathscr{S} \supseteq \mathscr{S}_{\mathrm{in}}$ which is consistent, by [CPS], Lemma 4.7 and Lemma 4.9. Consistency at the boundary follows by the convexity criterion Proposition 3.13. All walls added to obtain $\mathscr{S}$ are of the form

$$\Big((-\mathbb{R}_{\geq 0} m_0) \cap B, 1 + \sum_{\overline{m} \in \mathbb{R}_{>0} m_0} c_m z^m\Big).$$

Technically, this $\mathscr{S}$ is not quite a structure according to our definition because there might be distinct walls with the same support. However a standard structure can be obtained by replacing all walls with the same support with a single wall whose attached function is a product. Using this structure, we can build a family $\mathfrak{X}_k = \operatorname{Proj} S_k$ over the ring $\Bbbk[Q]/I_k$ with $I_k = (t^{k+1})$ for each k. Taking the inverse limit of the S_k gives a graded ring $\widehat{S}$, getting a projective family $\mathcal{X} \to T$ with

$$T = \operatorname{Spec} \varprojlim \Bbbk[Q]/I_k = \operatorname{Spec} \Bbbk[\![t]\!].$$

We sketch the properties of this family.

For $k = 0$, $\mathfrak{X}_0$ is a union of three toric varieties, each a weighted projective space. To see what the generic fibre of $\mathcal{X} \to T$ is, let us analyze local models near the vertices of B. Without loss of generality, consider the vertex $v_1 = (-1,-1)$. Our construction involves gluing the spectra of various rings. Explicitly, the ring R_{ρ_1} given by (2.17) is

$$R_{\rho_1} = (\Bbbk[Q]/I_k)[\Lambda_{\rho_1}][Z_+, Z_-]/(Z_+Z_- - (1 + tx^{-1}y^{-1})f_{\rho_1}t),$$

where

$$f_{\rho_1} = \prod_{\substack{(\mathfrak{d}, f_\mathfrak{d}) \in \mathscr{S} \setminus \mathscr{S}_{\mathrm{in}} \\ \mathfrak{d} = \rho_1}} f_\mathfrak{d}.$$

We also have two rings $R_{\sigma_\pm, \rho_\pm}$ where $\sigma_\pm$ are the two two-cells containing ρ_1 and $\rho_\pm$ are the corresponding edges of B containing v_1. Taking ρ_+ to have vertices v_1 and v_2, we have by (2.6)

$$R_{\sigma_+, \rho_+} = (\Bbbk[Q]/I_k)[x, y^{\pm 1}], \quad R_{\sigma_-, \rho_-} = (\Bbbk[Q]/I_k)[x^{\pm 1}, y].$$

We glue $\operatorname{Spec} R_{\rho_1}$ and $\operatorname{Spec} R_{\sigma_+, \rho_+}$ by localizing at Z_- and x respectively, and then identifying the generator $(1, 1)$ of Λ_{ρ_1} with xy and Z_+ with y. We glue $\operatorname{Spec} R_{\rho_1}$ and $\operatorname{Spec} R_{\sigma_-, \rho_-}$ by localizing at Z_+ and y respectively, and then identifying the generator $(1, 1)$ of Λ_{ρ_1} with xy and Z_- with y^{-1}.

It is easy to check that the ring of regular functions on this glued scheme can then be written as

$$R^k_{v_1} := (\Bbbk[Q]/I_k)[X, Y, W]/(XY + tW(1 + tW^{-1}) f_{\rho_1}(t, W)).$$

The inclusion of this ring in R_{ρ_1} is given by

$$\begin{aligned} X &\mapsto z^{(1,1)} Z_- \quad (z^{(1,1)} \in \Bbbk[\Lambda_\rho]) \\ Y &\mapsto Z_+ \\ W &\mapsto z^{(1,1)}. \end{aligned}$$

Note that the ideal is generated by $XY + t(W + t) f_{\rho_1}(t, W)$ and f_{ρ_1} is congruent to 1 modulo t.

The scheme $\operatorname{Spec} R^k_{v_1}$ is the affine completion of this glued scheme. There are similar descriptions of rings $R^k_{v_2}$, $R^k_{v_3}$, and the three schemes $\operatorname{Spec} R^k_{v_i}$ cover $\mathfrak{X}_k \setminus \{0\}$, where 0 is the point in $\mathfrak{X}_0$ where the three irreducible components meet.

The boundary $\mathfrak{D}_k$ of $\operatorname{Spec} R^k_{v_1} \subseteq \mathfrak{X}_k$ is given by $W = 0$, see Remark 2.18. Thus we see that our construction gives a family of pairs $(\mathcal{X}, \mathcal{D}) \to S$, and locally the equation for $\mathcal{D}$ to order k is $XY + t^2(1 + t(\cdots)) = 0$, clearly a smoothing of $XY = 0$. Thus $\mathcal{D} \to T$ is a smoothing of a triangle of $\mathbb{P}^1$'s.

We next claim that with $\eta = \operatorname{Spec} \Bbbk((t))$, the generic fibre $\mathcal{X}_\eta$ of $f : \mathcal{X} \to T$ is smooth. We sketch the argument. For sufficiently large k, it is easy to check that the singular locus of the map $f_k : \operatorname{Spec} R^k_{v_i} \to \operatorname{Spec} \Bbbk[Q]/I_k$ is not scheme-theoretically surjective in the sense of [GHK1], Definition-Lemma 4.1 and following. Furthermore, the argument of §4 of [GHK1] shows a similar statement for a neighbourhood of 0 in $\mathfrak{X}_k$. Thus the singular locus of $\mathfrak{X}_k \to \operatorname{Spec} \Bbbk[Q]/I_k$ is not scheme-theoretically surjective for large k. Now consider the singular locus of f. The formation of singular locus commutes with base change (see [GHK1], Definition-Lemma 4.1 again). As $\mathcal{X} \times_T \operatorname{Spec} \Bbbk[Q]/I_k = \mathfrak{X}_k$ for any k, we must not have $\mathrm{Sing}(f)$ surjecting onto T. Since f is proper, this means $\mathrm{Sing}(f)$ is disjoint from $\mathcal{X}_\eta$, and the latter scheme is smooth over η.

To identify $\mathcal{X}_\eta$, we proceed as follows. Note that the relatively ample line bundle $\mathcal{L}$ on $\mathcal{X}$ restricts to a very ample line bundle on $\mathfrak{X}_0$ by inspection, embedding $\mathfrak{X}_0$ as a surface of degree 9 in $\mathbb{P}^9$. This implies $\mathcal{L}|_{\mathcal{X}_\eta}$ is also very ample. Note also that $\mathcal{L}|_{\mathfrak{X}_0} \cong \omega^{-1}_{\mathfrak{X}_0}$, again by inspection. As $H^1(\mathfrak{X}_0, \mathcal{O}_{\mathfrak{X}_0})$ can be calculated to be 0, we also have $\mathcal{L}|_{\mathcal{X}_\eta} \cong \omega^{-1}_{\mathcal{X}_\eta}$. Thus, passing to $\bar\eta = \operatorname{Spec} \overline{\Bbbk((t))}$, we see $\mathcal{X}_{\bar\eta}$ is a del Pezzo surface of degree 9. Here $\overline{\Bbbk((t))}$ denotes the algebraic closure of $\Bbbk((t))$. From the classification of del Pezzo surfaces, we have $\mathcal{X}_{\bar\eta} \cong \mathbb{P}^2_{\bar\eta}$. So $\mathcal{X}_\eta$ is a Brauer-Severi scheme over η.

A result of Witt (see e.g., [GiSz], Corollary 6.3.7) implies that if $\Bbbk$ is an algebraically closed field of characteristic zero, then the Brauer group of $\Bbbk((t))$ is trivial. Thus $\mathcal{X}_\eta \cong \mathbb{P}^2_\eta$.

There remains the question of describing the theta functions we have constructed on $\mathbb{P}^2_\eta$. We do not see at this point how to describe these functions completely. The structure $\mathscr{S}$ is expected to be very complicated, containing non-trivial rays of every rational slope. Hence it is likely to be very difficult to control jagged paths. Nevertheless, there is a certain amount of symmetry which gives us some information.

There is an action of the group $H = \mathbb{Z}_3^2$, generated by α and β, on the data of our construction. The generator α acts on $B \subseteq \mathbb{R}^2$ as the linear transformation $\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$. This action preserves φ, and we then get an action on monomials lifting the action on $\mathbb{R}^2$. In particular, this gives an action on walls taking $(\mathfrak{p}, 1 + \sum c_m z^m)$ to $(\alpha(\mathfrak{p}), 1 + \sum c_m z^{\alpha(m)})$. One sees that $\mathscr{S}_{\mathrm{in}}$ is preserved by this action, and hence so is $\mathscr{S}$. Further, for $\tau \in \mathscr{P}$ of codimension 0 or 1, the ring R_τ is canonically identified using α with $R_{\alpha(\tau)}$. Thus α acts as an automorphism of $\mathfrak{X}_k / \operatorname{Spec}(\Bbbk[Q]/I_k)$.

The generator β leaves B fixed, but acts on monomials via

$$\beta(x) = \zeta x, \quad \beta(y) = \zeta^2 y, \quad \beta(t) = t,$$

where ζ is a primitive third root of unity. Again $\mathscr{S}_{\mathrm{in}}$ is left invariant under this action, so the same is true of $\mathscr{S}$. Then β also acts as automorphisms of the rings R_τ, hence again β induces an automorphism of $\mathfrak{X}_k / \operatorname{Spec}(\Bbbk[Q]/I_k)$.

In conclusion, the group H acts on $\mathfrak{X}_k / \operatorname{Spec}(\Bbbk[Q]/I_k)$ for all k (with the trivial action on $\operatorname{Spec} \Bbbk[Q]/I_k$) and hence acts on $\mathcal{X}/T$. This action preserves $\mathcal{D}$, and is clearly non-trivial on $\mathcal{D}$ since it permutes the components of $\mathcal{D}_0$.

Note furthermore that $\mathcal{D}_\eta$ has a point over η: certainly $\mathcal{D}_0$ has many $\Bbbk$-valued points which are non-singular points of $\mathcal{D}_0$, so by Hensel's lemma $\mathcal{D} \to \operatorname{Spec} \Bbbk[\![t]\!]$

has a section. Thus $\mathcal{D}_\eta$ has a $\Bbbk((t))$-valued point, and hence has the structure of an abelian variety.

In particular, if $\phi \in H$ induces an automorphism $\phi : \mathcal{D}_\eta \to \mathcal{D}_\eta$, then $\phi^*\mathcal{O}_{\mathbb{P}^2}(1)|_{\mathcal{D}_\eta} \cong \mathcal{O}_{\mathbb{P}^2}(1)|_{\mathcal{D}_\eta}$. Since the j-invariant of $\mathcal{D}_\eta$ is non-constant, the only choice for such an automorphism is translation by an element in the kernel of the polarization $\mathcal{O}_{\mathbb{P}^2}(1)|_{\mathcal{D}_\eta}$. It is then standard (see e.g., [Mu1]) that the group action of H lifts to an action on $\mathcal{O}_{\mathbb{P}^2}(1)|_{\mathcal{D}_\eta}$ after passing to a central extension

$$1 \to \mathbb{G}_m \to G \to H \to 0.$$

Here G is the Heisenberg group. The representation of H on $H^0(\mathcal{D}_\eta, \mathcal{O}_{\mathbb{P}^2}(1)|_{\mathcal{D}_\eta}) = \Bbbk((t))^{\oplus 3}$ is then isomorphic to the Schrödinger representation, that is, there is a basis x_0, x_1, x_2 (with indices taken modulo 3) of $H^0(\mathcal{D}_\eta, \mathcal{O}_{\mathbb{P}^2}(1)|_{\mathcal{D}_\eta})$ with (lifts of) α and β acting by

$$\alpha(x_i) = x_{i+1}, \quad \beta(x_i) = \zeta^i x_i$$

for $\zeta \in \Bbbk$ a primitive third root of unity.

Now consider the theta function ϑ_0 corresponding to $0 \in B(\mathbb{Z})$. Because the monomial corresponding to 0 is left invariant by G and the scattering diagram itself is invariant under the action of G, it follows that ϑ_0 is invariant. Since $\vartheta_0|_{\mathcal{X}_\eta}$ is a section of $\omega^{-1}_{\mathcal{X}_\eta} \cong \mathcal{O}_{\mathbb{P}^2}(3)$, ϑ_0 must be an invariant cubic. In the coordinates x_0, x_1, x_2 in which the Schrödinger representation is described above, the general such invariant cubic is $\lambda_1(x_0^3 + x_1^3 + x_2^3) + \lambda_2 x_0 x_1 x_2$. Here $\lambda_1, \lambda_2 \in \Bbbk((t))$. It is also easy to see that ϑ_0 vanishes on $\mathcal{D}$. Thus the equation of $\mathcal{D}_\eta$ is given by the above cubic, for some choice of λ_1, λ_2, and ϑ_0 takes the same form. Moreover, $\mathcal{D}$ is the result of applying our construction to ∂B with its decomposition into nine unit intervals. In dimension one our construction is purely toric and, for $B = S^1$, produces a Tate curve, with known j-invariant. Hence the quotient λ_1/λ_2 can be computed from this j-invariant of $\mathcal{D}$.

If $p \in B(\mathbb{Z}) \setminus \{0\}$, then one can classify jagged paths for p which are contained entirely in ∂B. Indeed, because of the form of $\mathscr{S}$, a jagged path which starts at $p \in \partial B$ can only bend at a ray of $\mathscr{S} \setminus \mathscr{S}_{\mathrm{in}}$ if it bends outwards. Thus jagged paths contained in ∂B can only bend at the rays of $\mathscr{S}_{\mathrm{in}}$, and a simple calculation shows that such jagged paths must bend as much as possible whenever a ray of $\mathscr{S}_{\mathrm{in}}$ is crossed. Using this, one can compare $\vartheta_p|_{\mathcal{D}}$ with theta functions on $\mathcal{D}$. One finds the description as given in the earlier part of this section. We omit the details.

Example 6.3. Consider the family $\mathcal{X} \to S$ of quartic K3 surfaces in $\mathbb{P}^3 \times S$ given by the equation

$$s(x_0^4 + x_1^4 + x_2^4 + x_3^4) + x_0 x_1 x_2 x_3 = 0, \tag{6.6}$$

where S is the spectrum of a discrete valuation ring over a field $\Bbbk$ with uniformizing parameter s. This is a toric degeneration (see Example 4.2 in [GrSi2]). If we use the polarization given by $\mathcal{O}_{\mathbb{P}^3}(1)|_{\mathcal{X}}$, we obtain an intersection complex $(B, \mathscr{P})$ which is a union of four standard simplices, forming a tetrahedron. There is in fact an affine structure with singularities on B which extends across the vertices. As in [GrSi2], this is specified by defining a *fan structure* at each vertex, i.e., an identification of a neighbourhood of each vertex with the neighbourhood of 0 of a fan. (See [GrSi1], Example 2.10 for the dual intersetion complex version of this example.) The fan structure at each vertex is given by the fan for $\mathbb{P}^2$, as the degeneration is normal crossings at the zero-dimensional strata of $\mathcal{X}_0$. Using $(B, \mathscr{P})$, we can work backwards and construct a smoothing of $\mathcal{X}_0$ using the algorithm of [GrSi4] to construct a consistent structure. The initial data used to construct this structure is induced by the log structure on $\mathcal{X}_0$ coming from the inclusion $\mathcal{X}_0 \subseteq \mathcal{X}$. Note that B has six singularities, one each at the barycenter of each edge. There are then initial walls emanating from each singular point, with attached function of the form $1 + z^{4m}$, where m is primitive with $\overline{m}$ tangent to the edge. We omit the details.

From this initial data, [GrSi4] gives a consistent structure, giving a polarized deformation $\mathfrak{X} \to \operatorname{Spec} \Bbbk[\![t]\!]$ as usual, with $\mathcal{L}$ the line bundle on $\mathfrak{X}$. For a simpler exposition of this result in two dimensions, see [Gr3], Chapter 6. Then $B(\mathbb{Z})$ consists of the vertices of $\mathscr{P}$, giving four theta functions $\vartheta_0, \ldots, \vartheta_3$ which are sections of $\mathcal{L}$. By construction, these can be chosen so their restriction to the central fibre gives $x_0, \ldots, x_3$. Since $\mathcal{L}$ is very ample when restricted to the central fibre, it is also very ample when restricted to the generic fibre $\mathfrak{X}_\eta$. It is then clear that $\mathcal{L}|_{\mathfrak{X}_\eta}$ embeds $\mathfrak{X}_\eta$ as a quartic surface in $\mathbb{P}^3_\eta$, so one can ask which quartic equation is satisfied by the ϑ_i's.

To see this, one can observe as in Example 6.2 that $\mathfrak{X}$ has a large symmetry group. First observe that B has an action of a permutation of the vertices, and this action preserves the initial structure, and hence preserves the structure defining $\mathfrak{X}$. This action acts on the theta functions ϑ_i by permutation also.

We can also find an action of multiplication by fourth roots of unity. Indeed, one can easily check that the monodromy of Λ around each singular point of B takes the form $\begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}$ in a suitable basis. Thus the local system $\Lambda \otimes_{\mathbb{Z}} \mathbb{Z}/4\mathbb{Z}$ has no monodromy, and hence is trivial. Fix an isomorphism $\Lambda \otimes_{\mathbb{Z}} \mathbb{Z}/4\mathbb{Z}$ with the constant sheaf with stalk $(\mathbb{Z}/4\mathbb{Z})^2$; this can be done by fixing an isomorphism $\Lambda_x \cong \mathbb{Z}^2$ at some point $x \in B_0$. In particular, given any character $\chi : (\mathbb{Z}/4\mathbb{Z})^2 \to \Bbbk^\times$, we obtain a map $\chi : \mathcal{P} \to \Bbbk^\times$ via the factorization $\mathcal{P} \to \Lambda \to \Lambda \otimes_{\mathbb{Z}} (\mathbb{Z}/4\mathbb{Z}) \cong (\mathbb{Z}/4\mathbb{Z})^2 \to \Bbbk^\times$. Thus such a χ gives a well-defined action on monomials. Because of the form of the initial walls of the structure, the structure is left invariant under this action, as are all relevant rings and gluing maps. Hence χ acts on $\mathfrak{X}$.

If we take H to be the group $S_4 \times \mathrm{Hom}((\mathbb{Z}/4\mathbb{Z})^2, \Bbbk^\times)$, then there is a central extension

$$1 \to \mathbb{G}_m \to G \to H \to 1$$

of H which acts on the line bundle $\mathcal{L}$ and hence on its space of sections. It is easy to see that there is a lift of an element $\alpha \in S_4$ to G acting by the corresponding permutation on the ϑ_i.

Given a character χ, it is clear that the theta functions are also eigensections of the action of a lift of χ to G. Continuing to write such a lift as χ, since we can modify a lift by an element of $\mathbb{G}_m$, we can always assume $\chi(\vartheta_0) = \vartheta_0$. Once this is fixed, the action of χ on the other ϑ_i is determined. To see this explicitly, let z^{m_i} be a monomial appearing in ϑ_i as expanded at the point v_0, the point of $B(\mathbb{Z})$ corresponding to ϑ_0. In particular, each m_i lives in the stalk of $\tilde{\mathcal{P}}_1$ at v_0, and the difference $m_i - m_0$ lives in the stalk of $\mathcal{P}$ at v_0. After taking the image of the difference in $\Lambda \otimes_{\mathbb{Z}} (\mathbb{Z}/4\mathbb{Z})$, we get a well-defined element of $\Lambda_{v_0} \otimes_{\mathbb{Z}} (\mathbb{Z}/4\mathbb{Z})$ only depending on i and not on the particular terms taken. Using the fan structure defining the affine structure in a neighbourhood of v_0, we can choose coordinates so that $v_0, \ldots, v_3$ have coordinates $(0,0)$, $(1,0)$, $(0,1)$ and $(-1,-1)$ respectively. From this one sees using these coordinates that the lifted action of χ is now

$$\chi(\vartheta_0) = \vartheta_0, \quad \chi(\vartheta_1) = \chi(1,0)\cdot\vartheta_1, \quad \chi(\vartheta_2) = \chi(0,1)\cdot\vartheta_2, \quad \chi(\vartheta_3) = \chi(-1,-1)\cdot\vartheta_3.$$

As the sections ϑ_i embed $\mathfrak{X}$ into $\mathbb{P}^3_{\Bbbk[\![t]\!]}$, and at $t = 0$ they satisfy the quartic equation $\vartheta_0\vartheta_1\vartheta_2\vartheta_3 = 0$, we obtain a family of quartics invariant under the action of G described above, and hence is necessarily of the form

$$\lambda(t)(\vartheta_0^4 + \vartheta_1^4 + \vartheta_2^4 + \vartheta_3^4) + \vartheta_0\vartheta_1\vartheta_2\vartheta_3 = 0, \tag{6.7}$$

with $\lambda(t)$ a formal power series in t vanishing at $t = 0$. In particular, the family is the base-change of an algebraic family, even if $\lambda(t)$ is not algebraic, and the theta functions are just the coordinates. That theta functions should have such a simple expression in general is not clear at all.

For $\Bbbk = \mathbb{C}$ one can also show that $\lambda(t)$ is analytic in t as follows. In the analytic version of the Dwork family (6.6) there are families of 2-cycles $\alpha = \alpha(s)$ and $\beta = \beta(s)$ with $g(s) = \exp(\mathrm{int}_\beta\Omega/\mathrm{int}_\alpha\Omega)$ an analytic coordinate on the parametrizing disc. Here Ω is a choice of holomorphic 2-form. These period integrals are of the form treated in [RS], hence can be computed also on $\mathfrak{X}$ to give a monomial in t. Comparing with (6.7) shows that $g(\lambda(t)) = c \cdot t$ for some $c \in \mathbb{C}^\times$. Then λ is obtained by inverting g.

Appendix A. The GS case

The purpose of this section is to discuss previous work of the first and last authors within the framework established in this paper. The main references are [GrSi2] and [GrSi4].

A.1. **One-parameter families.** The setup in [GrSi2] and [GrSi4] is more restrictive than here in that the affine structure extends over a neighbourhood of each vertex. In fact, the discriminant locus consists only of those codimension two cells of the barycentric subdivision neither containing vertices nor intersecting the interiors of maximal cells. For clarity we denote this smaller discriminant locus by $\breve{\Delta}$, and by $\iota : \breve{\Delta} \to B$ the inclusion. The strongest results in [GrSi2] and [GrSi4] are proved under the assumption that the affine singularities of B are *positive* and *simple*. Positivity is a necessary condition for the affine structure to come from a degeneration. Simplicity is a strong local primitivity condition expressed by requiring that certain integral polytopes spanned by the local monodromy vectors are elementary simplices[15]. In the positive, simple case over an algebraically closed field $\Bbbk$, one of the main results of [GrSi2] shows that the set of isomorphism classes of possible X_0 as log spaces over the standard log point is canonically in bijection with $H^1(B, \iota_*\check{\Lambda} \otimes \underline{\Bbbk}^\times)$ ([GrSi2], Theorem 5.4)[16]. This cohomology group provides so-called *lifted gluing data* $\mathbf{s} = (s_{\omega\tau})$ ([GrSi2], Definition 5.1), which induces both open gluing data and a log structure on $X_0 = X_0(\mathbf{s})$ over the standard log point $(\operatorname{Spec} \Bbbk, \Bbbk^\times \oplus \mathbb{N})$. Unlike in Subsection 5.2, the open gluing data obtained in this way consists of homomorphisms[17] $s_{\omega\tau} : \Gamma(W_{\omega\tau}, \iota_*\Lambda) \to \Bbbk^\times$ for *all* inclusions $\omega \to \tau$, regardless of the dimensions. Here $W_{\omega\tau}$ is the open star of the edge $\omega\tau$ in the barycentric subdivision as introduced in Subsection 5.2. Lifted gluing data also induces closed gluing data in the sense considered here. The log structure is equivalent to providing slab functions $f_{\rho,v}$ for any pair $\rho \in \mathscr{P}^{[n-1]}$ and $v \in \rho$ a vertex labelling a connected component of $\rho \setminus \breve{\Delta}$. There is an additional multiplicative compatibility condition for each $\tau \in \mathscr{P}^{[n-2]}$ that involves all $f_{\rho,v}$ with $\rho \supset \tau$, see also the discussion in [GrSi4], §1.2.

Assuming B bounded and with positive and simple singularities, the algorithm in [GrSi4] then readily produces a mutually compatible series of consistent wall structures $\mathscr{S}^{\mathrm{GS}} = \mathscr{S}_k$ on $(B, \mathscr{P})$ for $Q = \mathbb{N}$, $A = \Bbbk$ and $I = (t^{k+1}) \subseteq \Bbbk[t] = A[Q]$, for any $k \in \mathbb{N}$, see [GrSi4], Theorem 3.1. Here X_0 has an implicit dependence on

[15] An elementary simplex is a lattice simplex whose only integral points are its vertices.

[16] In [GrSi2] we mostly work with the *dual* intersection complex or fan picture, hence the dualization in the sheaf compared to the original statement, see [GrSi2], Proposition 1.50. A summary of the setup in the intersection picture of the present paper is contained in [GrSi4], §1.

[17] The notation in [GrSi4] is s_e for $e : \omega \to \tau$

the gluing data $\mathbf{s}$. The notion of wall structure is almost identical, there being two differences. The first is that Δ in [GrSi4] was chosen transverse to any rational polyhedral subset. In particular, our present requirement $\operatorname{Int}\mathfrak{b}\cap\Delta=\emptyset$ for all slabs $\mathfrak{b}$ can only be fulfilled for $\breve\Delta=\emptyset$. However, this requirement of transversality with Δ was purely technical and can be removed as follws. Modifying the argument in [GrSi4], Remark 5.3, consider $B\times[0,1]$ as an affine manifold with a discriminant locus restricting on $B\times\{0\}$ to the barycentrically centered one from the present setup and fulfilling the conditions required in [GrSi4] on $B\times(0,1]$. Because there is never any scattering on the boundary, the algorithm still works in this setup. Once the discriminant locus is barycentric, we can then decompose each slab $\mathfrak{b}$ into the closures of connected components of $\mathfrak{b}\setminus\Delta$. We then have the polyhedra of a wall structure $\mathscr{S}$ in the sense of Definition 2.11,2. For clarity we write $\underline{\mathfrak{b}}$ for the slabs in the sense of this paper obtained by decomposition. For a wall $\mathfrak{p}\in\mathscr{S}$ of codimension zero take the attached function $f_\mathfrak{p}$ identical to the one in $\mathscr{S}^{GS}$.

The different treatment of gluing data in the present work compared to [GrSi4] requires a modification of the slab functions as follows. Given the choice of open gluing data $\mathbf{s}=(s_{\omega\tau})$, one obtains open gluing data in the sense of §5.2 by taking[18] $s_{\sigma\underline{\rho}}=s_{\rho\sigma}^{-1}$. Then given a slab $\underline{\mathfrak{b}}$ in $\mathscr{S}$, the function $f_{\underline{\mathfrak{b}}}$ is obtained by choosing any point $x\in\operatorname{Int}\underline{\mathfrak{b}}$ and considering the function $f_{\mathfrak{b},x}$ attached to the point $x\in\mathfrak{b}$, where $\mathfrak{b}$ is the slab of $\mathscr{S}^{\mathrm{GS}}$ containing $\underline{\mathfrak{b}}$. Let v be the vertex of ρ contained in the same connected component of $\rho\setminus\breve\Delta$ as x. We then take

$$f_{\underline{\mathfrak{b}}}=s_{v\rho}^{-1}(f_{\mathfrak{b},x}). \tag{A.1}$$

This formula for $f_{\underline{\mathfrak{b}}}$ arises from the change of chambers homomorphisms θ of log rings in the case $\sigma_\mathfrak{u}\neq\sigma_{\mathfrak{u}'}$, see [GrSi4], lower half of p.1349.[19] Then [GrSi4], (1.11) implies (5.3).

For later use we observe that the order zero reduction $f_{\underline{\rho}}$ of $f_{\underline{\mathfrak{b}}}$ for any slab $\underline{\mathfrak{b}}\subseteq\underline{\rho}$ can be written down explicitly and turns out to have constant coefficients, not depending on gluing data. To state this result recall that in the present case with positive and simple singularities, the set

$$\Delta(\rho,v)=\big\{m_{\underline{\rho}\underline{\rho}'}\,\big|\,\underline{\rho}'\subseteq\rho\big\}$$

of monodromy vectors of closed paths in $W_\rho\setminus\Delta$ starting and ending at v, are the vertices of an elementary simplex. Here $\underline{\rho}\subseteq\rho$ is the cell of $\tilde{\mathscr{P}}$ containing v.

[18]The inverse arises from the different sign convention taken in [GrSi2], [GrSi4].

[19]The factor $D(s_{v\rho},\rho,v)$ from [GrSi4], Definition 1.20 does not appear here since we work with lifted gluing data.

Lemma A.1. *Let $(B,\mathscr{P})$ be bounded and with positive and simple singularities and $\underline{\mathfrak{b}}$ a decomposed slab. Then the order zero reduction of $f_{\underline{\mathfrak{b}}}$ is*

$$f_{\underline{\rho}}=\sum_{m\in\Delta(\rho,v)} z^m.$$

Proof. Let $v\in\underline{\mathfrak{b}}$ be the unique vertex of $\mathscr{P}$ contained in $\underline{\mathfrak{b}}$. Theorem 5.2,2 from [GrSi2] says that in the positive, simple case, the order zero slab functions $f_{v\to\rho}$ are determined uniquely by the gluing data. An explicit formula is given in the proof of Theorem 5.2,(2) on p.304 of [GrSi2]:

$$f_{v\to\rho}=\sum_{m\in\Delta(\rho,v)} s_{v\rho}(m)z^m.$$

Here z^m denotes the unique monomial of order zero with tangent vector m. The claimed formula is now immediate by reducing (A.1) modulo t. □

We continue with fitting the construction of [GrSi4] into the present framework.

Lemma A.2. *The wall structure $\mathscr{S}$ coming from* [GrSi4]*, Theorem 3.1 is consistent in the sense of Definition 3.9.*

Proof. The notion of consistency agrees in codimension zero, but differs in codimensions one and two. In codimension one, we first observe that our ring $R_{\underline{\mathfrak{b}}}$ arises as a fibre product of the rings R_{σ_+} and R_{σ_-} over R_ρ. This is discussed in the proof of [GrSi4], Lemma 2.34. Expressed in terms of this fibre product, the notion of consistency along a codimension one joint of [GrSi4], Definition 2.28, yields the notion in Definition 2.14.

Consistency in codimension two for $\mathscr{S}$ in the sense of Definition 3.9 is the content of [CPS], Proposition 3.2. □

Proposition A.3. *Let $\mathfrak{X}^{\mathrm{GS}}\to\operatorname{Spec}\big(\Bbbk[t]/(t^{k+1})\big)$ be the flat deformation constructed from $\mathscr{S}$ in* [GrSi4]*, §2.6. Then the complement of the codimension two strata of $X_0\subseteq\mathfrak{X}^{\mathrm{GS}}$ is canonically isomorphic to $\mathfrak{X}^\circ$ constructed in §2.4.*

Moreover, if the lifted gluing data **s** *is projective (Definition 5.13) then $\mathfrak{X}^{\mathrm{GS}}$ agrees with the projective scheme denoted by $\mathfrak{X}$ in Theorem 4.12.*

Proof. The constructions agree away from the codimension two locus, observing the partial gluing in codimension one discussed in the proof of Lemma A.2.

In the projective case, both $\mathcal{O}_{\mathfrak{X}^{\mathrm{GS}}}$ and $\mathcal{O}_{\mathfrak{X}}$ are sheaves on X_0 fulfilling Serre's condition S_2 and which are canonically isomorphic on X_0°, an open dense subset with complement of codimension two. Denote by $i:X_0^\circ\to X_0$ the inclusion. Then also $\mathfrak{X}=\mathfrak{X}^{\mathrm{GS}}$ canonically since

$$\mathcal{O}_{\mathfrak{X}}=i_*\mathcal{O}_{\mathfrak{X}^\circ}=i_*\mathcal{O}_{(\mathfrak{X}^{\mathrm{GS}})^\circ}=\mathcal{O}_{\mathfrak{X}^{\mathrm{GS}}},$$

by the S_2-condition. □

Remark A.4. In the projective setup Proposition A.3 provides a homogeneous coordinate ring S_k as a flat $\Bbbk[t]/(t^{k+1})$-algebra. Varying k, the S_k form an inverse system of $\Bbbk[\![t]\!]$-algebras. Thus taking the limit $S := \varprojlim S_k$ shows that in [GrSi4], Theorem 1.30, we do not only get a flat formal scheme over $\mathrm{Spf}(\Bbbk[\![t]\!])$, but a flat scheme $\mathfrak{X} := \mathrm{Proj}(S) \to \mathrm{Spec}(\Bbbk[\![t]\!])$, without imposing further cohomological assumptions and invoking Grothendieck's existence theorem as in [GrSi4], Corollary 1.31.

A similar remark holds in the case of higher dimensional bases discussed in §A.2.

A.2. **The universal formulation.** In [GrSi3], §5.2, it is discussed how to build a family $(X_0, \mathcal{M}_{X_0}) \to (\mathrm{Spec}\, A, \mathcal{M}_A)$ of toric log Calabi-Yau spaces ([GrSi2], Definition 4.3) parametrized by variations of lifted gluing data. The base is an algebraic torus with $A = \Bbbk\big[H^1(B, \iota_*\check\Lambda)^*\big] = \Bbbk\big[H^1(B, \iota_*\check\Lambda)^*_f\big]$ with $\Bbbk$ an algebraically closed field. Here the subscript f denotes the free part of a finitely generated abelian group. The set of closed points of this torus is $H^1(B, \iota_*\check\Lambda) \otimes \Bbbk^\times = H^1(B, \iota_*\check\Lambda)_f \otimes \Bbbk^\times$. The last equality is due to the fact that $\Bbbk^\times$ is a divisible group thanks to $\Bbbk$ being algebraically closed. The family depends on the choice of a right-inverse $\sigma_0 : H^1(B, \iota_*\check\Lambda)_f \to H^1(B, \iota_*\check\Lambda)$ of the quotient by the torsion subgroup[20] and on an element $s_0 \in H^1(B, \iota_*\check\Lambda \otimes \Bbbk^\times)$. The choice of s_0 selects one of the pairwise disjoint $H^1(B, \iota_*\check\Lambda) \otimes \Bbbk^\times$-torsors that cover $H^1(B, \iota_*\check\Lambda \otimes \Bbbk^\times)$ and which are parametrized by $H^2(B, \iota_*\Lambda)_{\mathrm{tors}}$, see [GrSi3], first displayed formula in §5.2. As a log scheme the base is the product of $\mathrm{Spec}\, A$ with trivial log structure and the standard log point. In particular, $\overline{\mathcal{M}}_A = \underline{\mathbb{N}}$ is constant and there is a global chart $\mathbb{N} \to \Gamma(\mathrm{Spec}\, A, \mathcal{M}_A)$. As a family of toric log Calabi-Yau spaces this family is defined by the pair (s_0, σ_0) viewed as an element in $H^1(B, \iota_*\check\Lambda \otimes \underline{A}^\times)$ as follows. Since A is a Laurent polynomial ring,

$$A^\times = \Bbbk^\times \oplus H^1(B, \iota_*\check\Lambda)^*_f.$$

Thus

$$H^1(B, \iota_*\check\Lambda \otimes \underline{A}^\times) = H^1(B, \iota_*\check\Lambda \otimes \underline{\Bbbk}^\times) \oplus \big(H^1(B, \iota_*\check\Lambda) \otimes H^1(B, \iota_*\check\Lambda)^*_f\big),$$

and the second summand equals $\mathrm{Hom}(H^1(B, \iota_*\check\Lambda)_f, H^1(B, \iota_*\check\Lambda))$. Thus (s_0, σ_0) can be viewed as an element of $H^1(B, \iota_*\check\Lambda \otimes \underline{A}^\times)$. Note that by compatibility

[20] The torsion subgroup of $H^1(B, \iota_*\check\Lambda)$ is related to isotrivial families. In fact, $H^1(B, \iota_*\check\Lambda \otimes \Bbbk[s^{\pm 1}]^\times)$ is isomorphic to $H^1(B, \iota_*\check\Lambda \otimes \Bbbk^\times) \oplus H^1(B, \iota_*\check\Lambda)$. Hence a class $\sigma_T \in H^1(B, \iota_*\check\Lambda)$ with $b \cdot \sigma_T = 0$ for some $b > 0$ defines gluing data $(1, \sigma_T)$ over $\mathbb{G}_m = \mathrm{Spec}\, \Bbbk[s^{\pm 1}]$ that becomes trivial after the base change $s \to s^b$. Similarly, our universal families for different choices of σ_0 become isomorphic after a finite étale cover.

with base change, the fibre of this family $(X_0, \mathcal{M}_{X_0}) \to (\operatorname{Spec} A, \mathcal{M}_A)$ over the closed point in $\operatorname{Spec} A$ defined by an element $\xi \in H^1(B, \iota_*\check{\Lambda})_f \otimes \Bbbk^\times$ is classified by $\sigma_0(\xi)\cdot s_0 \in H^1(B, \iota_*\check{\Lambda}\otimes \underline{\Bbbk}^\times)$.

Let us now refine the discussion in §A.1 to this situation. We will do this by first enlarging the log structure on $\operatorname{Spec} A$ to the universal one and then arguing that [GrSi4] also produces a consistent wall structure for this case.

In [GrSi2] and [GrSi4] the base monoid Q is always $\mathbb{N}$, but we have the freedom of choosing a strictly convex MPA-function φ with values in $\mathbb{N}$, which we fixed so far. We now want to replace $\mathbb{N}$ by the universal monoid analogous to Q_0 in Proposition 1.12. To work this out recall that the current choice of φ is more restrictive than in Definition 1.10 in that we require $\kappa_{\underline{\rho}}(\varphi) = \kappa_{\underline{\rho}'}(\varphi)$ for any $\underline{\rho}, \underline{\rho}'$ contained in the same $\rho \in \mathscr{P}^{[n-1]}$ and we impose additive conditions along codimension two cells, see Example 1.11,1. For a toric monoid Q' denote the subgroup of Q'^{gp}-valued MPA-functions in this restricted sense by $\mathrm{M\check{P}A}(B, Q'^{\mathrm{gp}}) \subseteq \mathrm{MPA}(B, Q'^{\mathrm{gp}})$ and let $\mathrm{M\check{P}A}(B, Q')$ be the corresponding monoid of convex functions. Refining Proposition 1.12, the universal point of view runs as follows. Since the additive condition is provided by a homomorphism into a torsion-free group, $\mathrm{MPA}(B,\mathbb{Z})/\mathrm{M\check{P}A}(B,\mathbb{Z})$ is torsion-free. As a consequence, the restriction map

$$r : \operatorname{Hom}\big(\mathrm{MPA}(B,\mathbb{N}),\mathbb{Z}\big) \longrightarrow \operatorname{Hom}\big(\mathrm{M\check{P}A}(B,\mathbb{N}),\mathbb{Z}\big)$$

is a surjection. Denote by $e_{\underline{\rho}_i}$ the generators of the monoid $Q_0 = \operatorname{Hom}\big(\mathrm{MPA}(B,\mathbb{N}),\mathbb{N}\big)$ as defined in Proposition 1.12. Then the kernel of r is the saturation of the subgroup generated by elements of the form (1) $e_{\underline{\rho}} - e_{\underline{\rho}'}$ for $\underline{\rho}, \underline{\rho}'$ both contained in the same codimension 1 cell ρ, and (2) elements of the form $\sum_i \langle m, n_i\rangle e_{\underline{\rho}_i}$. Here the $\underline{\rho}_i$, n_i are as in (1.7), and there is one element of the latter kind for each codimension two $\tau\in\mathscr{P}$, $\tau \not\subseteq \partial B$ and $m \in \Lambda_v$, $v\in\tau$ a vertex. Define Q to be the saturation of the submonoid of $\operatorname{Hom}\big(\mathrm{M\check{P}A}(B,\mathbb{N}),\mathbb{Z}\big)$ generated by $r(Q_0)$, so in particular we have a map $r: Q_0 \to Q$. Define also

$$\check{\varphi} := r\circ\varphi_0 \in \mathrm{M\check{P}A}(B,Q) \tag{A.2}$$

for $\varphi_0 \in \mathrm{MPA}(B, Q_0)$ the universal MPA-function from Proposition 1.12. In fact, by the definition of Q, the MPA-function $\check{\varphi}$ fulfills the properties defining $\mathrm{M\check{P}A}(B,Q)$ as a subspace of $\mathrm{MPA}(B,Q)$. Note also that by construction, $\check{\varphi}$ is convex. In [GrSi2], [GrSi4] we assume the existence of a strictly convex MPA-function with values in $\mathbb{N}$, and hence $\check{\varphi}$ is even strictly convex.

Analogously to φ_0 the MPA-function $\check{\varphi}$ has the universal property for restricted MPA-functions with values in any fine saturated monoid Q'. Indeed, given any Q'-valued restricted MPA-function ψ on B, we obtain a function $h: Q_0\to Q'$ so that

$\psi = h \circ \varphi_0$, by Proposition 1.12. On the other hand, because ψ is restricted, h must vanish on the elements of Q_0^{gp} generating the kernel of $r : Q_0^{\mathrm{gp}} \to Q^{\mathrm{gp}}$ described above, and hence h descends to a well-defined map $\breve{h} : r(Q_0) \to Q'$. Since Q' is fine and saturated, this map extends to $\breve{h} : Q \to Q'$. Then $\psi = h \circ \varphi_0 = \breve{h} \circ r \circ \varphi_0 = \breve{h} \circ \breve{\varphi}$ by construction. It is also not hard to see that if $\psi \in \mathrm{M\breve{P}A}(B, \mathbb{Z})$ takes values in integers, then the corresponding classifying map $\breve{h} : Q \to \mathbb{Z}$ is given by the homomorphism $\mathrm{M\breve{P}A}(B, \mathbb{Z})^* \to \mathbb{Z}$ that evaluates at ψ. Said differently, $\breve{\varphi}$ is the tautological restricted MPA-function with kinks $\kappa_\rho \in Q \subseteq \mathrm{M\breve{P}A}(B, \mathbb{Z})^*$ such that for any $\psi \in \mathrm{M\breve{P}A}(B, \mathbb{Z})$ we have

$$\kappa_\rho(\psi) = \langle \kappa_\rho(\breve{\varphi}), \psi \rangle \in \mathbb{Z}. \tag{A.3}$$

Construction A.5. (*Construction of* $\overline{\mathcal{M}}_{X_0}^{\breve{\varphi}}$.) Analogous to [GrSi2], Example 3.17, for the case of $\mathbb{N}$, there is a fine sheaf of monoids $\overline{\mathcal{M}}_{X_0}^{\breve{\varphi}}$ in the Zariski topology on X_0, with constant stalks along toric strata, along with a homomorphism $Q \to \Gamma(X_0, \overline{\mathcal{M}}_{X_0}^{\breve{\varphi}})$. For the construction observe that the affine structure on the $Q_{\mathbb{R}}^{\mathrm{gp}}$-torsor $\mathbb{B}_{\breve{\varphi}} \to B$ from Construction 1.14 now extends over the preimage of $\Delta \setminus \breve{\Delta}$. In particular, the sheaf of monomials $\mathcal{P}^+ \subseteq \mathcal{P} = \breve{\varphi}^* \Lambda_{\mathbb{B}_{\breve{\varphi}}}$ from §2.2 is defined over $B \setminus \breve{\Delta}$. Then the restriction of $\overline{\mathcal{M}}_{X_0}^{\breve{\varphi}}$ to the algebraic torus in the toric stratum $X_\tau \subseteq X_0$ is constant with stalks $\mathcal{P}_x^+ / \mathcal{P}_x^\times$ for any $x \in \operatorname{Int} \tau \setminus \Delta$. By the definition of $\mathrm{M\breve{P}A}(B, Q^{\mathrm{gp}}) \subseteq \mathrm{MPA}(B, Q^{\mathrm{gp}})$, local parallel transport yields canonical isomorphisms between the monoids $\mathcal{P}_x^+ / \mathcal{P}_x^\times$ for different choices of x, even between different connected components of $\tau \setminus \breve{\Delta}$. If $\omega \subseteq \tau$ and η_ω, η_τ are the generic points of the corresponding toric strata, the generization map $\overline{\mathcal{M}}_{X_0, \eta_\omega} \to \overline{\mathcal{M}}_{X_0, \eta_\tau}$ is defined by generization $\mathcal{P}_y^+ / \mathcal{P}_y^\times \to \mathcal{P}_x^+ / \mathcal{P}_x^\times$ for $x \in \operatorname{Int} \tau \setminus \breve{\Delta}$, $y \in \operatorname{Int} \omega \setminus \breve{\Delta}$. Since these generization maps are compatible with the map $Q \to \mathcal{P}_x^+ / \mathcal{P}_x^\times$, the sheaf $\overline{\mathcal{M}}_{X_0}^{\breve{\varphi}}$ comes with a homomorphism $Q \to \Gamma(X_0, \overline{\mathcal{M}}_{X_0}^{\breve{\varphi}})$.

Construction A.6. (*Construction of the log structure* $\mathcal{M}_{X_0}^{\breve{\varphi}}$ *on* X_0.) Recall that we now have two MPA-functions on B in the restricted sense, the universal one $\breve{\varphi}$ with values in Q and the chosen one φ with values in $\mathbb{N}$. By the universal property there is a unique homomorphism $\breve{h} : Q \to \mathbb{N}$ with $\varphi = \breve{h} \circ \breve{\varphi}$ inducing a homomorphism $\overline{\mathcal{M}}_{X_0}^{\breve{\varphi}} \to \overline{\mathcal{M}}_{X_0}$ of ghost sheaves on our family over the algebraic torus $\operatorname{Spec} A$. Since φ is strictly convex, $\breve{h}$ is a local homomorphism of monoids, that is, $\breve{h}^{-1}(0) = \{0\}$. Then

$$\mathcal{M}_{X_0}^{\breve{\varphi}} := \overline{\mathcal{M}}_{X_0}^{\breve{\varphi}} \times_{\overline{\mathcal{M}}_{X_0}} \mathcal{M}_{X_0} \tag{A.4}$$

is a sheaf of monoids, and the composition

$$\mathcal{M}_{X_0}^{\breve{\varphi}} \longrightarrow \mathcal{M}_{X_0} \longrightarrow \mathcal{O}_X$$

of the projection with the structure homomorphism for $\mathcal{M}_{X_0}$ defines a log structure on X_0 with ghost sheaf $\overline{\mathcal{M}}_{X_0}^{\breve\varphi}$. In fact, the preimage of $\mathcal{O}_{X_0}^\times$ in $\mathcal{M}_{X_0}^{\breve\varphi}$ is readily seen to be $\{0\}\times\mathcal{O}_{X_0}^\times\simeq\mathcal{O}_{X_0}^\times$.

Moreover, denote by $\mathcal{M}_A^Q$ the log structure on $\operatorname{Spec} A$ associated to the chart $Q\to A$ mapping $Q\setminus\{0\}$ to 0. Then the map from Q into sections of the first factor in (A.4) induces a morphism of log schemes

$$(X_0,\mathcal{M}_{X_0}^{\breve\varphi})\longrightarrow(\operatorname{Spec} A,\mathcal{M}_A^Q). \tag{A.5}$$

This morphism has a universal property for families of log schemes with closed fibres isomorphic to fibres of $X_0\to\operatorname{Spec} A$ and arbitrary log structures on the base. Since this is not important for the present discussion we omit the details.

Now we are in position to run the smoothing algorithm of [GrSi4] with the following modification. As ground field ($\Bbbk$ in [GrSi4]) take the quotient field $A_{(0)}$ of A. Denote by $I_0\subseteq A_{(0)}[Q]$ the ideal generated by $Q\setminus\{0\}=\breve h^{-1}\big(\mathbb{N}\setminus\{0\}\big)$. Then in the algorithm replace $\mathbb{N}$ by Q, but define the notion of order of exponents ([GrSi4], Definition 2.3) as before by first composing with $\breve h: Q\to\mathbb{N}$. Geometrically this corresponds to base changing from $A_{(0)}[Q]$ to $A_{(0)}[t]$ by means of $\breve h$. The change from $\mathbb{N}$ to Q enters in the propagation of exponents on B in the smoothing algorithm in that elements of Q are being added when changing cells. On a formal level this just means interpreting t^l as a monomial in $A_{(0)}[Q]$ rather than in $A_{(0)}[t]$. Moreover, by [GrSi4], Theorem 5.2, since the slab functions are defined over A, the smoothing algorithm nevertheless produces a wall structure defined over $A[Q]$. The result is a compatible system $\big(\mathscr{S}_k\big)_{k\in\mathbb{N}}$ of consistent wall structures, producing a compatible system of flat morphisms $\mathfrak{X}_k\to\operatorname{Spec}\big(A[Q]/I_0^k\big)$, or a flat formal scheme

$$\mathfrak{X}\longrightarrow\operatorname{Spf}\big(A[\![Q]\!]\big), \tag{A.6}$$

extending $X_0\to\operatorname{Spec} A$. Here $A[\![Q]\!]$ denotes the completion of $A[Q]$ with respect to $\breve h: Q\to\mathbb{N}$. In particular, the base of this family is the completion of the affine toric variety $\operatorname{Spec}\big(A[Q]\big)$ along its minimal toric stratum $\operatorname{Spec} A$.

To obtain a projective family, hence to make contact with Theorem 4.12, restrict to any closed subspace of $\operatorname{Spec} A$ with vanishing obstruction class $\mathrm{ob}_{\mathbb{P}}$ from Proposition 5.12 and Remark 5.14. To do this universally define an obstruction map $\mathrm{ob}_{\mathbb{P}}$ on lifted gluing data by composing the general obstruction map from (5.9), denoted $\mathrm{ob}_{\mathbb{P}}$ there, with the map turning lifted gluing data to closed gluing data:

$$\mathrm{ob}_{\mathbb{P}}: H^1(B,\iota_*\check\Lambda\otimes\underline{A}^\times)\longrightarrow H^1(B,\mathcal{Q}\otimes\underline{A}^\times)\longrightarrow H^2(B,A^\times).$$

With the previous noted equalities $A^\times=\Bbbk^\times\oplus H^1(B,\iota_*\check\Lambda)^*_f$ and $H^1(B,\iota_*\check\Lambda)^*=H^1(B,\iota_*\check\Lambda)^*_f$, and since the construction of $\mathrm{ob}_{\mathbb{P}}$ as a connecting homomorphism is functorial, this map decomposes as a direct sum of a map $\mathrm{ob}_{\mathbb{P}}^{\Bbbk^\times}: H^1(B, \iota_*\check{\Lambda} \otimes \underline{\Bbbk}^\times) \to H^2(B, \Bbbk^\times)$ and of

$$\mathrm{ob}_{\mathbb{P}}^{\mathbb{Z}} \otimes \mathrm{id} : H^1(B, \iota_*\check{\Lambda}) \otimes H^1(B, \iota_*\check{\Lambda})_f^* \longrightarrow H^2(B, \mathbb{Z}) \otimes H^1(B, \iota_*\check{\Lambda})_f^*. \tag{A.7}$$

Now take $s_0 \in H^1(B, \iota_*\check{\Lambda} \otimes \underline{\Bbbk}^\times)$ with $\mathrm{ob}_{\mathbb{P}}^{\Bbbk^\times}(s_0) = 0$ and let $K = \ker(\mathrm{ob}_{\mathbb{P}}^{\mathbb{Z}}) \subseteq H^1(B, \iota_*\check{\Lambda})$. Choose the splitting σ_0 of $H^1(B, \iota_*\check{\Lambda}) \to H^1(B, \iota_*\check{\Lambda})_f$ implicit in the above construction (denoted s_{id} in [GrSi3]) in such a way that it maps $K_f \subseteq H^1(B, \iota_*\check{\Lambda})_f$ into K, with K_f the free part of K. Then $\operatorname{Spec} A_{\mathbb{P}}$ with $A_{\mathbb{P}} := \Bbbk[K_f^*]$ is the maximal closed subspace of $\operatorname{Spec} A$ with vanishing $\mathrm{ob}_{\mathbb{P}}$. In fact, the restriction of the normalized gluing data to a closed subspace $\operatorname{Spec} \bar{A} \subseteq \operatorname{Spec} A$ is classified by the image of $(s_0, \sigma_0) \in H^1(B, \iota_*\check{\Lambda} \otimes \underline{A}^\times)$ in $H^1(B, \iota_*\check{\Lambda} \otimes \underline{\bar{A}}^\times)$. Since $\mathrm{ob}_{\mathbb{P}}^{\Bbbk^\times}(s_0) = 0$ one checks that in view of

$$H^1(B, \iota_*\check{\Lambda} \otimes \underline{A}_{\mathbb{P}}^\times) = H^1(B, \iota_*\check{\Lambda} \otimes \underline{\Bbbk}^\times) \oplus \big(H^1(B, \iota_*\check{\Lambda}) \otimes K_f^*\big)$$

the largest such $\bar{A}$ equals $\Bbbk[K_f^*]$.

According to Proposition 5.12 and Proposition 5.17 with ground ring $A_{\mathbb{P}}$ we can lift the wall structure to a consistent wall structure on $(\mathbf{C}B, \mathbf{C}\mathscr{P})$. In particular, Theorem 4.12 holds with $A_{\mathbb{P}}$ instead of A and with $I = I_0^k$ for any k. Taking the limit $k \longrightarrow \infty$ yields a family over $\operatorname{Spec}\big(A_{\mathbb{P}}[\![Q]\!]\big)$.

Theorem A.7. *Let $(B, \mathscr{P})$ be a bounded polyhedral manifold with positive and simple singularities admitting a strictly convex MPA function $\varphi \in \check{\mathrm{MPA}}(B, \mathbb{N})$. Denote by $Q \subseteq \check{\mathrm{MPA}}(B, \mathbb{N})^*$ the universal base monoid (see §A.2) and $A_{\mathbb{P}} = \Bbbk[K_f^*]$ with $K = \ker(\mathrm{ob}_{\mathbb{P}}^{\mathbb{Z}}) \subseteq H^1(B, \iota_*\check{\Lambda})$ as above.*

Then there exists a canonical polarized family

$$\mathfrak{X}_{\mathbb{P}} \longrightarrow \operatorname{Spec}(A_{\mathbb{P}}[\![Q]\!])$$

constructed from a consistent wall structure on $(\mathbf{C}B, \mathbf{C}\mathscr{P})$ and inducing the universal family of polarized log Calabi-Yau spaces over $\Bbbk$ with intersection complex $(B, \mathscr{P})$ (the polarized analogue of (A.5)*). Moreover, the statements of Theorem 4.12 apply verbatim.*

Proof. It only remains to check the statement on the family over $\operatorname{Spec} \Bbbk$, that is, after restriction to $\operatorname{Spec} A_{\mathbb{P}} \subseteq \operatorname{Spec}\big(A_{\mathbb{P}}[\![Q]\!]\big)$. First, working over $\Bbbk[\![Q]\!]$ rather than $\Bbbk[\![t]\!]$ only has the effect of incorporating a universal choice of ghost sheaf in the way discussed in Constructions A.5 and A.6. Thus it suffices to check the statement after pulling back to $\operatorname{Spec}\big(A_{\mathbb{P}}[\![t]\!]\big)$ by a homomorphism $h: Q \to \mathbb{N}$.

It is clear from comparison of the construction given here with the construction in [GrSi4] that $\mathfrak{X}_{\mathbb{P}}$ reduced modulo I_0 agrees with the base change $X_0^{\mathbb{P}} \to \operatorname{Spec} A_{\mathbb{P}}$ of $X_0 \to \operatorname{Spec} A$ from (A.5) outside codimension two, hence everywhere. Moreover, the log structure $\mathcal{M}'_{X_0}$ on X_0 coming with the construction in [GrSi4] agrees with the log structure $\mathcal{M}_{X_0}$ constructed in [GrSi3], §5.2. Finally, the algorithm in [GrSi4] also provides a compatible system of charts for the reduction of $\mathfrak{X}_{\mathbb{P}}$ modulo I^k and hence, by taking the limit $k \to \infty$, a system of charts for $\mathfrak{X}_{\mathbb{P}}$ compatible with charts for $(X_0, \mathcal{M}'_{X_0})$. Hence $\mathcal{M}'_{X_0}$ also agrees with the log structure on $X_0^{\mathbb{P}}$ defined by the closed embedding $X_0^{\mathbb{P}} \subseteq \mathfrak{X}_{\mathbb{P}}$. Thus we obtain the statement on the induced family over $\operatorname{Spec} A_{\mathbb{P}}$. □

As a final remark we give an interpretation of the obstruction map $\mathrm{ob}_{\mathbb{P}}^{\mathbb{Z}}$ in terms of the radiance obstruction.

Proposition A.8. *The obstruction map* $\mathrm{ob}_{\mathbb{P}}^{\mathbb{Z}} : H^1(B, \iota_*\check{\Lambda}) \to H^2(B, \mathbb{Z})$ *equals the cup product with the radiance obstruction* $c_B \in H^1(B, \iota_*\Lambda)$ *followed by the trace homomorphism.*

Proof. Since by [GrSi2], Proposition 1.29, the radiance obstruction c_B vanishes locally, it can be defined as the extension class $\tilde{c}_B \in \operatorname{Ext}^1(\iota_*\check{\Lambda}, \underline{\mathbb{Z}})$ of

$$0 \longrightarrow \underline{\mathbb{Z}} \longrightarrow \mathcal{A}\!f\!f(B, \mathbb{Z}) \longrightarrow \iota_*\check{\Lambda} \longrightarrow 0,$$

the push-forward to B of (1.1). The obstruction map $\mathrm{ob}_{\mathbb{P}}^{\mathbb{Z}}$ from (A.7) is a connecting homomorphism of the associated long exact cohomology sequence. The result now follows from the standard fact that connecting homomorphisms are given by cup product with the extension class. Identifying $H^1(B, \iota_*\check{\Lambda})$ with $\operatorname{Ext}^1_B(\underline{\mathbb{Z}}, \iota_*\check{\Lambda})$ this cup product is the Yoneda composition product,

$$\operatorname{Ext}^1_B(\iota_*\check{\Lambda}, \underline{\mathbb{Z}}) \otimes \operatorname{Ext}^1_B(\underline{\mathbb{Z}}, \iota_*\check{\Lambda}) \longrightarrow \operatorname{Ext}^2_B(\underline{\mathbb{Z}}, \underline{\mathbb{Z}}) = H^2(B, \mathbb{Z}).$$

In terms of sheaf cohomology this means taking the cup product with c_B to arrive at a class in $H^2(B, \iota_*\check{\Lambda} \otimes \iota_*\Lambda)$, followed by H^2 of the trace homomorphism $\iota_*\check{\Lambda} \otimes \iota_*\Lambda \to \underline{\mathbb{Z}}$. □

A.3. **Equivariance.** There are two tori acting compatibly on the universal families constructed in §A.2. To define these actions, denote by $\check{\mathrm{PL}}(B)$ the subgroup of $\mathrm{PL}(B) = \mathrm{PL}(B, \mathbb{Z})$ consisting of *restricted PL-functions*, that is, having kinks satisfying the conditions in codimensions one and two stated in Example 1.11,1. The space $\check{\mathrm{PL}}(B)$ agrees with the space of piecewise linear function in [GrSi2], Definition 1.43. Writing $\mathrm{M\check{P}A}(B) = \mathrm{M\check{P}A}(B, \mathbb{Z})$, the map $\kappa : \mathrm{PL}(B, \mathbb{Z}) \to \mathrm{MPA}(B, \mathbb{Z})$ associating to a PL-function φ the associated MPA-function with kinks $\kappa_\rho(\varphi)$, then descends to a map $\check{\mathrm{PL}}(B) \to \mathrm{M\check{P}A}(B)$ that we also denote κ.

Now the first action is by the relative torus from §4.4 with character lattice $\check{\mathrm{PL}}(B)^*$, treated abstractly in Theorem 4.17. This action is trivial on the coefficient ring A. The second torus has character lattice the dual of $\mathrm{M\check{P}A}(B)/\kappa\big(\check{\mathrm{PL}}(B)\big)$, up to finite index. This latter action is typically non-trivial on $A = \Bbbk[H^1(B, \iota_*\check{\Lambda})^*]$, which is graded by the dual of the connecting homomorphism $c_1 : \mathrm{M\breve{P}A}(B) \to H^1(B, \iota_*\check{\Lambda})$.

Let us first treat the relative torus action. Our universal family is a flat formal family $\mathfrak{X} \to \mathrm{Spf}(A[\![Q]\!])$ or, restricting to projective gluing data, a flat projective family $\mathfrak{X}_{\mathbb{P}} \to \mathrm{Spec}(A_{\mathbb{P}}[\![Q]\!])$. Here $Q \subseteq \mathrm{M\breve{P}A}(B,\mathbb{Z})^*$, $A = \Bbbk[H^1(B, \iota_*\check{\Lambda})^*_f]$ and $A_{\mathbb{P}} = \Bbbk[K^*_f]$ with $K = \ker(\mathrm{ob}^{\mathbb{Z}}_{\mathbb{P}})$, and K_f its free part, as discussed in §A.2. We are now in the situation with non-trivial gluing data commented on in Remark 4.19. Recall from Proposition 4.15 that the identity component of $\mathrm{Aut}_A(X_0)$ is the torus over A with character lattice $\mathrm{PL}(B)^*$. We are going to extend the restriction of this action to the subtorus with character lattice $\Gamma := \mathrm{\breve{P}L}(B)^*$ to $\mathfrak{X}$.[21]

In the notation of §4.4 we take the Γ-grading on A to be trivial, the map $\delta_B : \mathrm{PL}(B)^* \to \mathrm{\breve{P}L}(B)^*$ dual to the inclusion, and δ_Q the dual of the map $\mathrm{\breve{P}L}(B) \to \mathrm{M\breve{P}A}(B)$:

$$\delta_Q : Q \hookrightarrow \mathrm{M\breve{P}A}(B)^* \longrightarrow \mathrm{\breve{P}L}(B)^* = \Gamma.$$

Commutativity in the compatibility diagram (4.8) is trivially true. The degree zero part of $A[Q]$ is $A^Q_0 = A[Q']$ with $Q' = \delta_Q^{-1}(0)$. In particular, $A \subseteq A^Q_0$, and hence the gluing data have degree zero as required in Remark 4.19.

Proposition A.9. *The action of the torus* $\mathrm{Spec}(A[\Gamma]) \subseteq \mathrm{Aut}_A(X_0)$ *on* X_0 *extends canonically to actions on* $\mathfrak{X} \to \mathrm{Spf}(A[\![Q]\!])$ *and on* $\mathfrak{X}_{\mathbb{P}} \to \mathrm{Spec}(A_{\mathbb{P}}[\![Q]\!])$.

Proof. To apply Theorem 4.17, modified for non-trivial gluing data according to Remark 4.19, it remains to check that the wall structure $\mathscr{S}$ is homogeneous in the sense of Definition 4.16. We argue inductively and first observe that the initial wall structure $\mathscr{S}_0$ is homogeneous. Indeed, Lemma A.2, which continues to hold with universal gluing data with the same proof, shows that an initial slab function $f_{v\to\rho}$ is a sum of monomials $z^{m_{\underline{\rho}\underline{\rho}'}}$ with $m_{\underline{\rho}\underline{\rho}'}$ the local monodromy vectors from (1.3) and $\underline{\rho} \subseteq \rho$ the cell of $\tilde{\mathscr{P}}$ with $v \in \underline{\rho}$. But $\varphi \in \mathrm{PL}(B)$ lies in $\mathrm{\breve{P}L}(B)$ only if it is invariant under monodromy along ρ, which is equivalent to $\nabla_{m_{\underline{\rho}\underline{\rho}'}}(\varphi) = 0$ for all $\underline{\rho}' \subseteq \rho$. Here $\nabla_{m_{\underline{\rho}\underline{\rho}'}}$ is the directional derivative on PL-functions defined in (4.9). This shows $\deg_\Gamma(z^{m_{\underline{\rho}\underline{\rho}'}}) = 0$. Hence any order zero slab function $f_{v\to\rho}$ is homogeneous of degree 0.

Next observe that the wall crossing isomorphisms $\theta_{\mathfrak{p}}$ are of the form

$$z^m \longmapsto \tilde{f}_{\mathfrak{p}}^{\langle n_{\mathfrak{p}}, m\rangle} z^m,$$

with $\tilde{f}_{\mathfrak{p}}$ differing from $f_{\mathfrak{p}}$ by changing the constants via an application of gluing data, see the two displayed formulas on p.1349 of [GrSi4], Construction 2.24.

[21] Elements of $\mathrm{Aut}_A(X_0)$ not in this subtorus move the log-singular locus on the central fibre; hence their action on X_0 can not extend to an action on $\mathfrak{X}$ with trivial action on $\mathrm{Spec}\, A$.

Now assume inductively that the wall structure $\mathscr{S}_k$ at order k is homogeneous. Then up to subdivision of walls and slabs, the wall structure $\mathscr{S}_{k+1}$ at order $k+1$ is obtained from $\mathscr{S}_k$ by adding walls carrying monomials of order $k+1$ and changing the functions $f_{\underline{\mathfrak{b}}}$ carried by slabs $\underline{\mathfrak{b}}$ by adding monomials of order $k+1$. Consistency is checked by composing automorphisms associated to walls containing a local affine submanifold $\mathfrak{j}$ of codimension two (a *joint*) in a cyclic order. New walls are always created by such a consistency check of $\mathscr{S}_k$ in a version of the tropical vertex group [GPS] at order $k+1$, as in Lemma 3.7 in [GrSi4][22]. This shows that if θ is the composition of automorphisms for walls in $\mathscr{S}_k$ containing $\mathfrak{j}$, then to order $k+1$ there is a unique expansion

$$\theta = \exp\Big(\sum_i c_i z^{m_i}\partial_{n_i}\Big), \tag{A.8}$$

with $m_i \in \Lambda_x$, $n_i \in \check{\Lambda}_x$ for some $x \in \mathfrak{j}$ and c_i contained in (a localization of) $A[\Lambda_{\mathfrak{j}}]$ and the order of each summand along $\mathfrak{j}$ equal to $k+1$. Each summand on the right-hand side produces one new wall emanating from $\mathfrak{j}$. Now all the rings involved are graded, and by the induction hypothesis θ is homogeneous of degree zero. Moreover, in the computation of the exponential on the right-hand side any cross terms have order strictly larger than $k+1$ and therefore vanish. Hence omitting any summands $c_i z^{m_i}\partial_{n_i}$ with $c_i z^{m_i}$ of non-zero degree from the right-hand side of (A.8) leads to another expression of θ as the exponential of a vector field of order $k+1$. Thus by uniqueness of the expansion in (A.8), any of the monomials $c_i z^{m_i}$ of newly inserted walls are homogeneous of degree zero.

For joints contained in cells of codimensions one and two, we modify the slab function $f_{\underline{\mathfrak{b}}}$ with $\underline{\mathfrak{b}}$ contained in a codimension one cell $\rho \supseteq \mathfrak{j}$ by the addition of those terms $c_i z^{m_i}$ with $m_i \in \Lambda_\rho$. Thus the slab functions stay homogeneous of degree zero in the algorithm as well.

Apart from some subdivision of slabs or walls, which obviously does not spoil homogeneity, the insertion of new walls and the change of slab functions, there are two more changes to obtain $\mathscr{S}_{k+1}$. The first of these (Step II in the algorithm of [GrSi4]) is the propagation of changes made to one slab function $f_{\underline{\mathfrak{b}}}$ to other slabs $\underline{\mathfrak{b}}'$ contained in the same codimension one cell ρ, see [GrSi4], §3.5. This step only adds homogeneous terms $c_i z^{m_i}$ of degree zero from $f_{\underline{\mathfrak{b}}}$ to $f_{\underline{\mathfrak{b}}'}$. Hence this step also preserves homogeneity.

[22]If $\mathfrak{j}$ is contained in a cell of $\mathscr{P}$ of codimension one or two, the coefficient ring has to be enlarged from $A[\Lambda_{\mathfrak{j}}]$, the Laurent polynomial ring with exponents the integral tangent vectors to $\mathfrak{j}$, to a localization at the slab functions $f_{\underline{\rho}}$ with $\underline{\rho} \supseteq \mathfrak{j}$. The careful treatment of this situation in [GrSi4], Chapter 4, assures that no denominators appear in the coefficients of newly inserted walls.

The last modification (Step III in the algorithm of [GrSi4]) concerns the normalization condition by adding terms of the form $c_i z^{m_{\underline{\rho\rho'}}}$ with $c_i \in A[Q]$ to the slab functions ([GrSi4], §3.6). The coefficient c_i is again obtained from a unique expansion of a homogeneous expression by a process that is linear at the given order. Hence by the same argument as before each c_i is homogeneous of degree zero, as is $z^{m_{\underline{\rho\rho'}}}$. Thus this step also does not spoil homogeneity of the wall structure $\mathscr{S}_{k+1}$.

Note that homogeneity of $\mathscr{S}_{k+1}$ for all k establishes the torus action readily on $\mathfrak{X}$, not only on the complement of the codimension two locus $\mathfrak{X}^\circ$.

In the projective setup the homogeneous coordinate ring becomes naturally Γ-graded, hence the statement in this case. □

Remark A.10. The action of the relative torus on A being trivial implies that we have an induced action on the fibre over any closed point in $\operatorname{Spec} A$, viewed as a log space over the log point $(\operatorname{Spec} \Bbbk, Q \oplus \Bbbk^\times)$. This action keeps the isomorphism class of this log space over $(\operatorname{Spec} \Bbbk, Q \oplus \Bbbk^\times)$, in agreement with the interpretation of $\operatorname{Spec} A$ as a moduli space of log spaces over a log point.

This is in contrast to the second action below, which acts effectively on $\operatorname{Spec} A$.

The second torus, referred to as the *regluing torus*, has character lattice the dual of a finite index sublattice H of $c_1\big(\mathrm{M\check{P}A}(B)\big) \subseteq H^1(B, \iota_*\check\Lambda)$. Here $c_1 : \mathrm{M\check{P}A}(B) \to H^1(B, \iota_*\check\Lambda)$ is the connecting homomorphism coming from the analogue for restricted PL-functions of the short exact sequence (1.5) (see [GrSi2], Definition 1.45). The regluing torus acts by changing how $\mathfrak{X}$ is glued from open subsets. To define this action, we start with the exact sequence

$$\mathrm{P\check{L}}(B) \xrightarrow{\kappa} \mathrm{M\check{P}A}(B) \xrightarrow{c_1} H^1(B, \iota_*\check\Lambda).$$

Neither κ nor c_1 need to have saturated images. With the superscripts "sat" and "tor" denoting the saturation of a subgroup and the torsion part of an abelian group, define

$$\tilde H = c_1(\mathrm{M\check{P}A}(B)), \quad K := \kappa\big(\mathrm{P\check{L}}(B)\big)^{\mathrm{sat}} = c_1^{-1}\big(H^1(B, \iota_*\check\Lambda)^{\mathrm{tor}}\big),$$

and choose complements $\tilde L$ to $K \subseteq \mathrm{M\check{P}A}(B)$ and F to $\tilde H^{\mathrm{sat}} \subseteq H^1(B, \iota_*\check\Lambda)$. Then we have direct sum decompositions

$$\mathrm{M\check{P}A}(B) = K \oplus \tilde L, \quad H^1(B, \iota_*\check\Lambda) = \tilde H^{\mathrm{sat}} \oplus F,$$

and c_1 composed with the quotient by the torsion subgroup of $H^1(B, \iota_*\check\Lambda)$ maps $\tilde L$ isomorphically to $\tilde H_f = \tilde H / \tilde H^{\mathrm{tor}}$. The dual of the inclusion $\tilde H \subseteq \tilde H^{\mathrm{sat}}$ defines a finite index sublattice $(\tilde H^{\mathrm{sat}})^* \subseteq \tilde H^*$. The torus with character lattice $\tilde H^*$ acts canonically on the finite étale extension $\Bbbk[\tilde H^* \oplus F^*]$ of $A = \Bbbk[H^1(B, \iota_*\check\Lambda)^*] =$

$\Bbbk[(\tilde{H}^{\mathrm{sat}})^* \oplus F^*]$. If $H^1(B, \iota_*\check{\Lambda})$ is torsion-free this action lifts to the pull-back of the universal family.

In general, the action of the regluing torus depends on a good representative of the universal gluing data (s_0, σ_0) from §A.2 compatible with $c_1 : \mathrm{M\check{P}A}(B) \to H^1(B, \iota_*\check{\Lambda})$. To find this representative we may need another finite étale extension of base rings and go over to an isogeneous torus. Denote by $q_f : H^1(B, \iota_*\check{\Lambda}) \to H^1(B, \iota_*\check{\Lambda})_f$ the quotient by the torsion subgroup.

Lemma A.11. *There is a finite index sublattice $L \subseteq \tilde{L}$ with the property*

$$\sigma_0 \circ q_f|_{c_1(L)} = \mathrm{id}_{c_1(L)}\,.$$

Proof. This follows since σ_0 being a right-inverse to q_f, the image of $\sigma_0 \circ q_f - \mathrm{id}$ lies in the torsion subgroup of $H^1(B, \iota_*\check{\Lambda})$. □

Choose a finite index sublattice $L \subseteq \tilde{L}$ as given by the lemma and define $H = c_1(L) \subseteq \tilde{H}$. Then $\sigma_0|_{H_f}$ factors over $c_1|_L$ and $H \simeq H_f$ is torsion-free and isomorphic to L via c_1. Note that since $H^{\mathrm{sat}} = \tilde{H}^{\mathrm{sat}}$ we still have the direct sum decomposition

$$H^1(B, \iota_*\check{\Lambda}) = H^{\mathrm{sat}} \oplus F.$$

Since H and F are free we identify them with their respective images in $H^1(B, \iota_*\check{\Lambda})_f$. We take $L^* \simeq H^*$ as the character lattice of the regluing torus. Now $H \oplus F$ defines a sublattice of finite index in $H^1(B, \iota_*\check{\Lambda})$. Dualizing we obtain a finite étale extension of rings

$$A = \Bbbk[H^1(B, \iota_*\check{\Lambda})^*] = \Bbbk[(H^{\mathrm{sat}})^* \oplus F^*] \longrightarrow \tilde{A} := \Bbbk[H^* \oplus F^*].$$

The action of the regluing torus is only defined after pulling back the universal family $\mathfrak{X} \to \mathrm{Spf}\,\big(A[\![Q]\!]\big)$ from §A.2 by the finite and étale morphism $\mathrm{Spf}\,\big(\tilde{A}[\![Q]\!]\big) \to \mathrm{Spf}\,\big(A[\![Q]\!]\big)$.

For the projective case the alternative definition

$$\mathcal{M}\check{\mathcal{P}}\mathcal{A}(B, \mathbb{Z}) = \check{\mathcal{P}}\mathcal{A}(B, \mathbb{Z})/\mathcal{A}\mathit{ff}(B, \mathbb{Z})$$

shows that $c_1 : \mathrm{M\check{P}A}(B) \to H^1(B, \iota_*\check{\Lambda})$ factors over $H^1(B, \mathcal{A}\mathit{ff}(B, \mathbb{Z}))$. Thus since $\mathrm{ob}_{\mathbb{P}}^{\mathbb{Z}}$ is the connecting homomorphism of the long exact sequence for (1.1), pushed-forward to B by ι_*, we get $\mathrm{ob}_{\mathbb{P}}^{\mathbb{Z}} \circ c_1 = 0$. Hence $\mathrm{ob}_{\mathbb{P}}^{\mathbb{Z}}$ vanishes on H and we can choose $F \subseteq \ker \mathrm{ob}_{\mathbb{P}}^{\mathbb{Z}}$ as a complement to $H^{\mathrm{sat}} \subseteq \ker \mathrm{ob}_{\mathbb{P}}^{\mathbb{Z}}$ to obtain an analogous ring extension $A_{\mathbb{P}} \subseteq \tilde{A}_{\mathbb{P}}$. The resulting families are

$$\tilde{\mathfrak{X}} \longrightarrow \mathrm{Spf}\,\big(\tilde{A}[\![Q]\!]\big), \qquad \tilde{\mathfrak{X}}_{\mathbb{P}} \longrightarrow \mathrm{Spec}\,\big(\tilde{A}_{\mathbb{P}}[\![Q]\!]\big). \tag{A.9}$$

The action of the regluing torus on the base space of these families is defined by H^*-gradings on $\tilde{A}$ and on $Q \subseteq \mathrm{M\check{P}A}(B)^*$. We define the grading on Q by the transpose of the inclusion $L \subseteq \mathrm{M\check{P}A}(B,\mathbb{Z})$, that is, by the composition

$$\delta_Q : Q \longrightarrow \mathrm{M\check{P}A}(B)^* = K^* \oplus \tilde{L}^* \longrightarrow \tilde{L}^* \longrightarrow L^*. \tag{A.10}$$

Here the first arrow is the inclusion, the second arrow the projection, the third the transpose of $L \to \tilde{L}$. The grading on $\tilde{A}$ or $\tilde{A}_{\mathbb{P}}$ is defined by projection, followed by the isomorphism $H^* \simeq L^*$:

$$H^* \oplus F^* \longrightarrow H^* \xrightarrow{\simeq} L^*. \tag{A.11}$$

Next, recall the universal restricted MPA-function $\breve{\varphi} \in \mathrm{M\check{P}A}(B,Q)$ from Equation (A.2). Composing with δ_Q, that is, restricting each kink $\kappa_\rho(\breve{\varphi}) \in Q \subseteq \mathrm{M\check{P}A}(B,\mathbb{Z})^*$ to $L \subset \mathrm{M\check{P}A}(B,\mathbb{Z})$, yields an element

$$\breve{\varphi}|_L \in \mathrm{M\check{P}A}(B, L^*) = \mathrm{Hom}\big(L, \mathrm{M\check{P}A}(B,\mathbb{Z})\big).$$

It follows from (A.3) that this composition of $\breve{\varphi}$ with the restriction to L, that is, as a map

$$\breve{\varphi}|_L : L \longrightarrow \mathrm{M\check{P}A}(B,\mathbb{Z}),$$

is just the inclusion homomorphism[23]. Now let $\big(\breve{\varphi}_\tau\big)_{\tau\in\mathscr{P}}$ be a representative of $\breve{\varphi}$ and define

$$\psi_\tau = \breve{\varphi}_\tau|_L = \delta_Q \circ \breve{\varphi}_\tau.$$

Thus for any $\tau \in \mathscr{P}$ we now have a choice of PL-function $\psi_\tau : |\Sigma_\tau| \to \mathrm{Hom}(L,\mathbb{R})$ on the fan Σ_τ defined by the tangent wedges to τ, with kinks

$$\kappa_\rho(\psi_\tau) = \kappa_\rho(\breve{\varphi})|_L \in L^*,$$

for any $\rho \in \mathscr{P}^{[n-1]}$ containing τ. For convenience of the later discussion we take $\breve{\varphi}_\tau|_{\Lambda_\tau} = 0$.[24] These choices give the desired good representative of the gluing data σ_0 on $H = c_1(L)$. To make a precise statement, recall that B has an open cover $\mathscr{W}$ by open stars W_τ of the barycentric subdivision of $\mathscr{P}$ containing $\mathrm{Int}\,\tau$ ([GrSi2], Definition 1.25). This open cover is acyclic for $\iota_*\check{\Lambda}$ ([GrSi2], Lemma 5.5). Elements of $H^1(B,\iota_*\check{\Lambda})$ therefore can be represented by Čech 1-cocycles $\big(s_{\omega\tau}\big)_{\omega,\tau}$ with labelling by pairs $\omega,\tau \in \mathscr{P}$ with $\omega \subsetneq \tau$ and $s_{\omega\tau} \in \Gamma(W_\omega \cap W_\tau, \iota_*\check{\Lambda})$.

Lemma A.12. *The restriction of c_1 to $L \subseteq \mathrm{M\check{P}A}(B,\mathbb{Z})$,*

$$c_1|_L : L \longrightarrow H^1(B, \iota_*\check{\Lambda})$$

can be represented by the Čech 1-cocyle $\big(\sigma_{\omega\tau}\big)_{\omega,\tau} \in C^1(\mathscr{W}, \iota_*\check{\Lambda}) \otimes L^*$ with

$$\sigma_{\omega\tau} = \psi_\tau|_{W_\omega\cap W_\tau} - \psi_\omega|_{W_\omega\cap W_\tau}.$$

[23]Another way to put this is to observe that as an element of $\mathrm{M\check{P}A}(B, \mathrm{MPA}(B,\mathbb{Z})^*) = \mathrm{Hom}(\mathrm{M\check{P}A}(B,\mathbb{Z}), \mathrm{M\check{P}A}(B,\mathbb{Z}))$, the universal MPA-function $\breve{\varphi}$ is the identity. Restricting to L then gives the inclusion of L into $\mathrm{M\check{P}A}(B,\mathbb{Z})$.

[24]This choice has the effect that the action is trivial on order zero monomials with exponents tangent to the cell considered.

Proof. This is immediate from the fact that c_1 arises as a connecting homomorphism in the long exact cohomology sequence for the analogue of (1.5) (see [GrSi2], Definition 1.45). □

We are now ready to construct the action of the regluing torus.

Proposition A.13. *The actions of the torus* $\mathrm{Spec}\left(\Bbbk[L^*]\right)$ *on* $\mathrm{Spf}\left(\tilde{A}[\![Q]\!]\right)$ *and on* $\mathrm{Spec}\left(\tilde{A}_{\mathbb{P}}[\![Q]\!]\right)$ *defined by the gradings* (A.10),(A.11) *lift to actions on the finite pull-backs* $\tilde{\mathfrak{X}} \to \mathrm{Spf}\left(\tilde{A}[\![Q]\!]\right)$ *and* $\tilde{\mathfrak{X}}_{\mathbb{P}} \to \mathrm{Spec}\left(\tilde{A}_{\mathbb{P}}[\![Q]\!]\right)$ *in* (A.9) *of the universal families.*

Proof. We only discuss the case of $\tilde{\mathfrak{X}} \to \mathrm{Spf}\left(\tilde{A}[\![Q]\!]\right)$. The statement for the projective family then follows as in the proof of Proposition A.9 by grading the homogeneous coordinate ring.

Step 1: Choice of gluing data. By functoriality, the gluing data describing the central fibres of $\tilde{\mathfrak{X}} \to \mathrm{Spf}\left(\tilde{A}[\![Q]\!]\right)$ are given by the image of

$$(s_0,\sigma_0) \in H^1(B,\iota_*\check{\Lambda}\otimes \underline{A}^\times) = H^1(B,\iota_*\check{\Lambda}\otimes\Bbbk^\times)\oplus \mathrm{Hom}\left(H^1(B,\iota_*\check{\Lambda})_f, H^1(B,\iota_*\check{\Lambda})\right)$$

under

$$A^\times = \Bbbk^\times \oplus H^1(B,\iota_*\check{\Lambda})^*_f = \Bbbk^\times \oplus (H^{\mathrm{sat}})^* \oplus F^* \longrightarrow \tilde{A}^\times = \Bbbk^\times \oplus H^* \oplus F^*.$$

Thus this base change simply leaves s_0 unchanged and restricts σ_0 to $H\oplus F \subseteq H^1(B,\iota_*\check{\Lambda})_f = (H^{\mathrm{sat}})_f \oplus F$. Denote this restriction of σ_0 by $\tilde{\sigma}_0 : H\oplus F \to H^1(B,\iota_*\check{\Lambda})$. As a Čech 1-cocycle, $\tilde{\sigma}_0$ is represented by $(\tilde{\sigma}_{\omega\tau})_{\omega,\tau}$ with $\tilde{\sigma}_{\omega\tau} \in \Gamma(W_\omega\cap W_\tau,\iota_*\check{\Lambda})\otimes(H\oplus F)^*$.

Now by Lemma A.12 we can choose a representative $(\tilde{\sigma}_{\omega\tau})_{\omega,\tau}$ of $\tilde{\sigma}_0$ in such a way that

$$\tilde{\sigma}_{\omega\tau}\circ q_f\circ c_1 = \psi_\tau|_{W_\omega\cap W_\tau} - \psi_\omega|_{W_\omega\cap W_\tau} \in \Gamma(W_\omega\cap W_\tau,\iota_*\check{\Lambda})\otimes L^*. \tag{A.12}$$

We write $\left(\tilde{s}_{\omega\tau}\right)_{\omega,\tau}$ for the corresponding representative of the base-changed universal gluing data $(s_0,\tilde{\sigma}_0)$.

Step 2: Definition of the grading. We are now in position to define the grading on our rings. As in the proof of Proposition A.9 we prove inductively that the wall structure is homogeneous of degree 0. The grading on the coefficent ring $\tilde{A}[\![Q]\!]$ has already been defined in (A.10) and (A.11). In [GrSi4] the k-th order approximation to $\mathfrak{X}$ is glued from rings denoted $R^k_{\omega\to\tau,\sigma}$ indexed by cells $\omega\subseteq\tau$ in $\mathscr{P}$ and a reference maximal cell σ containing τ ([GrSi4], Construction 2.7). These $\tilde{A}[\![Q]\!]$-algebras are given by the localization at order zero slab functions $f_{v\to\rho}$ with $\rho\supseteq\omega$, of a quotient by a monomial ideal of a monomial ring. The monomials z^m have exponents $m\in P_{\omega,\sigma}$ with

$$P_{\omega,\sigma} = \left\{ m = (\overline{m},h)\in\Lambda_\sigma\oplus Q^{\mathrm{gp}} \,\middle|\, h\in\check{\varphi}_\omega(\overline{m})+Q\right\}.$$

Here $\breve{\varphi}_\omega$ is the representative of $\breve{\varphi}$ on W_ω chosen above, now viewed as a piecewise linear map $\Lambda_\sigma \to Q^{\mathrm{gp}}$ by means of a chart for the affine structure at any point of $\operatorname{Int}\omega$. This description is independent of the choice of point by the local monodromy invariance of our notion of piecewise linear function. The definition of $P_{\omega,\sigma}$ in [GrSi4], Construction 2.7 is more intrinsic, but the equivalence with the one given here is not hard to show.

For $m = (\overline{m}, h)$ we now define

$$\deg_{L^*} z^m = \delta_Q(h),$$

with δ_Q the grading on Q from (A.10). In particular, the degree of a monomial of order zero, that is $m = (\overline{m}, h)$ with $h = \breve{\varphi}_\omega(\overline{m})$, equals $\psi_\omega(\overline{m}) = \delta_Q\big(\breve{\varphi}_\omega(\overline{m})\big)$.

It is instructive to write down explicitly the action on the rings in codimension zero and one used otherwise in this paper. On the rings for chambers $R_{\mathfrak{u}} = R_\sigma$ the grading is trivial on the monomials by our choice $\breve{\varphi}_\sigma = 0$ and hence just comes from the grading on the coefficients in $\tilde{A}[Q] \subseteq \tilde{A}[\![Q]\!]$. For a slab $\underline{\mathfrak{b}}$ let ρ be the codimension one cell containing $\underline{\mathfrak{b}}$ and σ, σ' the two adjacent maximal cells, respectively. Then our ring $R_{\underline{\mathfrak{b}}}$ arises from the rings $R^k_{\omega\to\tau,\sigma}$ as a fibre product (see the proof of [GrSi4], Lemma 2.34):

$$R_{\underline{\mathfrak{b}}} = R^k_{\rho\to\sigma,\sigma} \times_{R^k_{\rho\to\rho,\sigma}} R^k_{\rho\to\sigma',\sigma'}. \tag{A.13}$$

The homomorphisms in this fibre product are defined by the relevant changes of strata and change of chambers ([GrSi4], Construction 2.24). Since $\breve{\varphi}_\rho|_{\Lambda_\rho} = 0$ the induced grading of the ring $R_{\underline{\mathfrak{b}}}$ is trivial on the monomials z^m with $m \in \Lambda_\rho$. For the two remaining generators we have

$$\deg_{L^*} Z_+ = \psi_\rho(\xi) \in L^*, \quad \deg_{L^*} Z_- = \psi_\rho(-\xi) \in L^*.$$

Recall that our sign conventions say that Z_+ maps to a monomial on the maximal cell $\sigma = \sigma(\rho)$, see (2.14). Provided the slab function $f_{\underline{\mathfrak{b}}}$ is homogeneous of degree zero, this definition of the grading of monomials turns $R_{\underline{\mathfrak{b}}}$ into a graded $\tilde{A}[\![Q]\!]$-algebra. In fact, the only relation $Z_+Z_- - f_{\underline{\mathfrak{b}}} t^{\kappa_\rho}$ is homogeneous of degree $\delta_Q(\kappa_\rho) = \psi_\rho(\xi) + \psi_\rho(-\xi) \in L^*$.

Step 3: Homogeneity of order zero slab functions. Lemma A.1 continues to hold for universal gluing data with the same proof. Thus the order zero reduction of one of our slab functions $f_{\underline{\mathfrak{b}}}$ equals $f_{\underline{\rho}} = \sum_{m\in\Delta(\rho,v)} z^m$. The degree $\deg_{L^*}(z^m)$ of any monomial occuring in $f_{\underline{\rho}}$ equals $\psi_\rho(\overline{m})$ with $m = m_{\underline{\rho\rho'}}$. Now as a restricted PL-function, ψ_ρ is invariant under monodromy in $W_\rho \setminus \Delta$ and hence $\psi_\rho(\overline{m}) = 0$ for any $m \in \Delta(\rho, v)$. Note this argument holds regardless of our special choice of PL-functions ψ_ρ. This shows that the order zero reduction of a slab function $f_{\underline{\mathfrak{b}}}$ is homogeneous of degree zero, as an element of the ring $R^0_{\rho\to\rho,\sigma}$ for $\rho \supset \underline{\mathfrak{b}}$.

Homogeneity of the slab functions in the other rings $R^0_{\omega\to\tau,\sigma}$ then follows by homogeneity of the gluing morphisms of type (I) in the following Step 4.

Step 4: Homogeneity of gluing. We have to check homogeneity of the gluing morphisms in [GrSi4], Construction 2.24. We assume inductively that the functions associated to slabs and walls are homogeneous of degree zero. There are three cases.

(I) (Change of strata). For cells $\omega \subseteq \omega' \subseteq \tau' \subseteq \tau \subseteq \sigma$ with σ maximal, we have a homomorphism[25]

$$R^k_{\omega\to\tau,\sigma} \longrightarrow R^k_{\omega'\to\tau',\sigma}, \quad z^m \longmapsto \tilde{s}_{\omega\omega'}(m)^{-1}\cdot z^m. \tag{A.14}$$

Assuming without loss of generality that z^m has order zero, as an element of $R^k_{\omega\to\tau,\sigma}$ we have $\deg_{L^*} z^m = \psi_\omega(\overline{m})$. Hence homogeneity follows from computing the degree of the right-hand side of (A.14) in $R^k_{\omega'\to\tau',\sigma}$:

$$\deg_{L^*}\big(\tilde{s}_{\omega\omega'}(m)^{-1}\cdot z^m\big) = \deg_{L^*}\big(-\tilde{\sigma}_{\omega\omega'}(\overline{m})\big)+\psi_{\omega'}(\overline{m}) \;=\; -\sigma_{\omega\omega'}(\overline{m})+\psi_{\omega'}(\overline{m}) = \psi_\omega(\overline{m}).$$

For the last equality see Lemma A.12.

(II.1) (Change of chambers $\mathfrak{u}\to\mathfrak{u}'$ with $\mathfrak{u},\mathfrak{u}'\subseteq\sigma$). This is a sequence of wall crossing homomorphisms, each defined by $\tilde{s}_{\omega\sigma}(f_\mathfrak{p})$ with $f_\mathfrak{p}$ homogeneous of degree zero. Thus $f_\mathfrak{p}$ can be written as a sum of expressions $c_m z^m$ with $c_m\in\tilde{A}[Q]$ and z^m a monomial of order zero on σ. Now it holds $\deg_{L^*} z^m = 0$ by our choice of $\breve{\phi}_\sigma$, and then necessarily also $\deg_{L^*} c_m = 0$. As an element of $R^k_{\omega\to\tau,\sigma}$, the term $c_m\tilde{s}_{\omega\sigma}(m)z^m$ has then degree

$$\deg_{L^*}\big(c_m\tilde{s}_{\omega\sigma}(m)z^m\big) = \sigma_{\omega\sigma}(\overline{m}) + \psi_\omega(\overline{m}) = \psi_\sigma(\overline{m}) = 0.$$

(II.2) (Change of chambers $\mathfrak{u}\to\mathfrak{u}'$ with $\mathfrak{u}\subseteq\sigma$, $\mathfrak{u}'\subseteq\sigma'$ and $\sigma\neq\sigma'$). This is the most interesting case $\theta: R^k_{\omega\to\tau,\sigma}\to R^k_{\omega\to\tau,\sigma'}$ of two chambers $\mathfrak{u}\subseteq\sigma$, $\mathfrak{u}'\subseteq\sigma'$ separated by a slab $\underline{\mathfrak{b}}\subseteq\rho$. Now by (I) we have a homogeneous epimorphism $R^k_{\omega\to\rho,\sigma}\to R^k_{\omega\to\tau,\sigma}$ and the ring $R^k_{\omega\to\rho,\sigma}$ is a localization with a product of degree zero slab functions of $R^k_{\rho\to\rho,\sigma}$, see [GrSi4], Remark 2.9. The analogous statement holds for σ exchanged by σ'. Thus it is enough to treat the case $\omega=\tau=\rho$ and prove homogeneity of the change of chambers homomorphism $R^k_{\rho\to\rho,\sigma}\to R^k_{\rho\to\rho,\sigma'}$.

All order zero monomials z^m of the two rings with $\overline{m}\in\Lambda_\rho$ have degree zero and are mapped to each other by identifying Λ_ρ as sublattices of Λ_σ and $\Lambda_{\sigma'}$ via parallel transport through $\rho\setminus\breve{\Delta}$. In particular, θ maps the localizing elements to each other and hence is also homogeneous on the localized subrings generated

[25] The formulas in [GrSi4] define homomorphisms of log rings; here we only need and state the induced homorphisms of ordinary rings. Moreover, since we work with lifted gluing data, the restriction of gluing data to a maximal cell necessary in [GrSi4] is not needed here.

by Λ_ρ. Each of the two rings has two more generators $z_+, z_- \in R^k_{\rho\to\rho,\sigma}$ and $z'_+, z'_- \in R^k_{\rho\to\rho,\sigma'}$ with homogeneous relations

$$z_+z_- = t^{\kappa_\rho}, \quad z'_+z'_- = t^{\kappa_\rho}.$$

The change of chamber homomorphism also depends on the choice of a vertex $v \in \omega \subseteq \rho$, which selects one of the codimension one cells $\underline{\rho} \subseteq \rho$ of the barycentric subdivision. Recall also the choice of primitive normal vector $\xi = \xi(\rho) \in \Lambda_\sigma$ for $\sigma = \sigma(\rho)$ and denote by $\xi' \in \Lambda_{\sigma'}$ the parallel transport of ξ to σ' through $\operatorname{Int}\underline{\rho}$. For easy comparison with the rings $R_{\underline{\mathfrak{b}}}$ with $\underline{\mathfrak{b}} = \mathfrak{b} \cap \underline{\rho}$ the decomposed slab from §A.1, we choose $z_+ = z^{(\xi,\check\varphi_\rho(\xi))}$ and $z'_+ = z^{(\xi',\check\varphi_\rho(\xi))}$. According to [GrSi4], p.1349, and (A.1) we have

$$\begin{aligned} \theta(z_+) &= f_{\underline{\mathfrak{b}}}\cdot (z'_-)^{-1}\cdot t^{\kappa_\rho} &&= f_{\underline{\mathfrak{b}}}\cdot z'_+ \\ \theta(z_-) &= f_{\underline{\mathfrak{b}}}^{-1}\cdot (z'_+)^{-1}\cdot t^{\kappa_\rho} &&= f_{\underline{\mathfrak{b}}}^{-1}\cdot z'_-. \end{aligned}$$

Now $\deg_{L^*} z_+ = \deg_{L^*} z'_+ = \psi_\rho(\xi)$ and $f_{\underline{\mathfrak{b}}}$ is homogeneous of degree zero, and similarly for z_- This shows homogeneity of θ.

This wall crossing homomorphism induces the grading on the ring $R_{\underline{\mathfrak{b}}}$ already discussed in Step 2.

Step 5: Homogeneity of wall structure. Assuming inductively that the wall structure $\mathscr{S}_k$ at order k is homogeneous of degree zero, following the steps in the algorithm of [GrSi4] shows homogeneity of the wall structure $\mathscr{S}_{k+1}$ at the next order, just as in the proof of Proposition A.9. □

Remark A.14. It is obvious from the constructions that our two group actions commute. Thus we have an action of a product torus with character lattice $\check{\mathrm{PL}}(B)^*\oplus L^*$ on the finite étale pull-backs $\tilde{\mathfrak{X}} \to \operatorname{Spf}\big(\tilde A[\![Q]\!]\big)$ and $\tilde{\mathfrak{X}}_{\mathbb{P}} \to \operatorname{Spec}\big(\tilde A_{\mathbb{P}}[\![Q]\!]\big)$ of the universal families. Here $L \subseteq \mathrm{M}\check{\mathrm{P}}\mathrm{A}(B)$ together with $\kappa\big(\check{\mathrm{PL}}(B)\big)$ spans a finite index subgroup of $\mathrm{M}\check{\mathrm{P}}\mathrm{A}(B)$. The first torus factor contains the subtorus with character lattice $H^0(B,\iota_*\check\Lambda)$ dealing with automorphisms of the family relative the base. The quotient by this torus has character lattice $\mathrm{M}\check{\mathrm{P}}\mathrm{A}(B)^* = Q^{\mathrm{gp}}$, up to finite index. Hence up to isogeny it can be identified with the torus for the affine toric variety $\operatorname{Spec}\Bbbk[Q]$. Only the second torus acts non-trivially on the fibre $\operatorname{Spec}(\tilde A)$ of $\operatorname{Spf}\big(\tilde A[\![Q]\!]\big) \to \operatorname{Spf}\big(\Bbbk[\![Q]\!]\big)$ over 0. This fibre parametrizes log structures relative to the standard log point and hence only this part induces a non-trivial action on $H^1(B,\iota_*\check\Lambda\otimes\Bbbk^*)$.

This action of the full-dimensional subtorus

$$\operatorname{Spec}\big(\Bbbk[\kappa(\check{\mathrm{PL}})\oplus L^*]\big) \subseteq \operatorname{Spec}\big(\Bbbk[\mathrm{M}\check{\mathrm{P}}\mathrm{A}(B,\mathbb{Z})^*]\big)$$

on the central fibre as a log space over the log point $(\operatorname{Spec}\Bbbk, Q\otimes\Bbbk^\times)$ simply acts by changing the chart of the log point. Thus this action is trivial when viewed as acting on a morphism of abstract log spaces, but not when viewed as acting on log spaces over $(\operatorname{Spec}\Bbbk, Q\otimes\Bbbk^\times)$.

A.4. **The non-simple case in two dimensions.** In two dimensions the local rigidity assumption in [GrSi4] is empty. We can therefore also treat non-simple singularities. The singularities are then at the barycenters of edges with local affine monodromy conjugate to $\left(\begin{smallmatrix}1&0\\ r&1\end{smallmatrix}\right)$. Here the edge is parallel to the first coordinate axis and $r\ge 1$. We call such a singularity an *r-fold singularity*, so $r=1$ is a simple singularity.

The following proposition generalizes [GrSi2], Theorem 5.4 to the non-simple case in two dimensions, formulated in the Legendre dual fan picture as in loc.cit., thus not requiring any projectivity assumption.

Proposition A.15. *Let $(\check B,\check{\mathscr P})$ be a closed two-dimensional polyhedral affine manifold with positive singularities (at the barycenters of the edges). Denote by n_r the number of r-fold singularities and let $K=\sum_r n_r(r-1)$.*

Then the set of isomorphism classes of positive log Calabi-Yau spaces over the standard log point $(\operatorname{Spec}\Bbbk,\mathbb N)$ *with dual intersection complex* $(\check B,\check{\mathscr P})$ *modulo isomorphism preserving $\check B$ is $H^1(\check B,\iota_*\Lambda\otimes\Bbbk^\times)\times\Bbbk^K$.*

Proof. Let $(X_0,\mathcal M_{X_0})$ be a log Calabi-Yau space as in the statement. Reexamination of the proof of [GrSi2], Theorem 5.4, shows that there exists a unique isomorphism class of lifted gluing data $\check s\in H^1(\check B,\iota_*\Lambda\otimes\Bbbk^\times)$ normalizing $(X_0,\mathcal M_{X_0})$ ([GrSi2], Definition 4.24). Unlike the simple case ([GrSi2], Theorem 5.2,2), now the normalization condition does not fix the log structure completely. Let $\rho\in\check{\mathscr P}$ be an edge containing an r-fold singularity and $f_{\rho,v}\in\mathbb C[w]$ the slab function determining the log structure on the toric stratum $X_\rho\subseteq X_0$ associated to ρ. Here $v\in\rho$ is a choice of vertex and w is the toric coordinate for $X_\rho\simeq\mathbb P^1$ at the corresponding zero-dimensional stratum. Then $\deg f_{\rho,v}=r$ and as in the proof of [GrSi2], Theorem 5.2,2, the normalization condition determines the constant and highest coefficients of $f_{\rho,v}$. Thus $f_{\rho,v}=1+a_1w+\ldots+a_{r-1}w^{r-1}+c(\check s)w^r$ with $c(\check s)\in\Bbbk^\times$ determined by the gluing data. The other coefficients $a_2,\ldots,a_{r-1}$ are completely free to vary, contributing a factor $\Bbbk^{r-1}$.

Once a choice of vertex on each edge with a singularity has been made, this description is unique up to changing $\check s$ by a coboundary. □

With the generalization to the non-simple case from Proposition A.15, the previous discussion of the case of simple singularities generalizes with the only change of replacing $A_{\mathbb P}=\Bbbk[K_f^*]$ by $\tilde A_{\mathbb P}=A_{\mathbb P}[\mathbb N^K]=\Bbbk[K_f^*\oplus\mathbb N^K]$. In particular, the additional factor $\mathbb N^K$ does not affect projectivity, the obstruction $\mathrm{ob}_{\mathbb P}^{\mathbb Z}$ still only takes the first factor $H^1(B,\iota_*\check\Lambda)$ as an input.

An interesting additional feature is the preservation of singularities in the family. To state this fact, denote by $\mathscr{S}_k$ the inductively obtained wall structure on $(B,\mathscr{P})$ that is consistent modulo I_0^k. Then for $\rho\in\mathscr{P}^{[1]}$ let $f_\rho\in\tilde{A}_{\mathbb{P}}[w^{\pm 1}]$ be the order zero slab function in $\mathscr{S}_k$ for the stratum ρ as given by the codimension one case of Definition 2.11,1. Here w is the toric coordinate along the stratum $X_\rho\subseteq X_0$. Then f_ρ equals the reduction modulo I_0 of any slab function $f_{\underline{\mathfrak{b}}}$ with $\underline{\mathfrak{b}}\subseteq\rho$ and it is related to $f_{\rho,v}$ in the proof of Proposition A.15 via formula (A.1).

Theorem A.16. *Let $(B,\mathscr{P})$ be a closed two-dimensional polyhedral affine manifold with positive singularities and assume there exists a strictly convex $\varphi\in$* $\mathrm{M\check{P}A}(B,\mathbb{N})$. *Then the conclusions of Theorem A.7 hold with $A_{\mathbb{P}}$ replaced by $\tilde{A}_{\mathbb{P}}=A_{\mathbb{P}}[\mathbb{N}^K]$, yielding a family*

$$\mathfrak{X}_{\mathbb{P}}\longrightarrow\operatorname{Spec}(A_{\mathbb{P}}[\![Q]\!])\times_{\Bbbk}\mathbb{A}^K_{\Bbbk}.$$

Moreover, the completion of $\mathfrak{X}_{\mathbb{P}}$ along the big cell of the toric stratum for an edge $\rho\in\mathscr{P}^{[1]}$ is isomorphic relative $\tilde{A}_{\mathbb{P}}[\![Q]\!]$ to

$$\operatorname{Spf}\big(\tilde{A}_{\mathbb{P}}[\![Q]\!][x,y,w^{\pm1}]/(xy-f_\rho z^{\kappa_\rho})\big)$$

In particular, for a closed point $(s,a)\in\operatorname{Spec}(A_{\mathbb{P}})\times\mathbb{A}^K_{\Bbbk}$ with $f_\rho(s,a)$ having n_r^ρ zeros of order r, the generic fibre of the restriction of $\mathfrak{X}_{\mathbb{P}}$ to $\{(s,a)\}\times\operatorname{Spec}\Bbbk[\![Q]\!]$ has n_r^ρ many A_{r-1} singularities with closure intersecting $X_\rho\subseteq X_0$.

Proof. Recall that Legendre duality ([GrSi2], §1.4) swaps the fan and cone pictures ([GrSi2], Theorem 2.34) and transforms an r-fold singularity along an edge ρ into an r-fold singularity along the Legendre-dual edge ([GrSi2], Proposition 1.50,1). Thus Proposition A.15 applies to the Legendre dual of $(B,\mathscr{P},\varphi)$ for some choice of strictly convex $\varphi\in\mathrm{M\check{P}A}(B,\mathbb{N})$.

The local model for the order k deformation $\mathfrak{X}^\circ_{\mathbb{P}}$ along the codimension one stratum $X_\rho\subseteq X_0$ is $Z_+Z_-=f_{\underline{\mathfrak{b}}}z^{\kappa_\rho}$ in $\tilde{A}_{\mathbb{P}}[Q]/(I_0^{k+1})[Z_+,Z_-,w^{\pm1}]$. Now by the inductive construction, at any finite order a slab function $f_{\underline{\mathfrak{b}}}$ for $\underline{\mathfrak{b}}\subseteq\rho$ is of the form $f_{\underline{\mathfrak{b}}}=f_\rho\cdot\prod_\mu(1+a_\mu w^{l_\mu})$ with $a_\mu\in I_0$. At order k the slab $\underline{\mathfrak{b}}$ only factors with $a_\mu\in I_0^k$ are being added. Thus taking the limit $k\to\infty$ for slabs $\underline{\mathfrak{b}}_k$ containing a general point of ρ, this product converges to some $f_{\underline{\mathfrak{b}}}=f_\rho\cdot h\in\tilde{A}_{\mathbb{P}}[\![Q]\!][w^{\pm1}]$. Now take $x=Z_+$, $y=h^{-1}Z_-$ to arrive at the stated equation for the formal completion along $X^\circ_\rho\subseteq\mathfrak{X}^\circ_{\mathbb{P}}$. □

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DPMMS, Centre for Mathematical Sciences, Wilberforce Road, Cambridge, CB3 0WB, UK
E-mail address: mgross@dpmms.cam.ac.uk

Department of Mathematics and Statistics, University of Massachusetts, Amherst, MA01003-9305
E-mail address: hacking@math.umass.edu

FB Mathematik, Universität Hamburg, Bundesstrasse 55, 20146 Hamburg, Germany
E-mail address: siebert@math.uni-hamburg.de