Compact moduli of plane curves

Paul Hacking

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Abstract

We construct a compactification $\mathcal{M}_d$ of the moduli space of plane curves of degree d. We regard a plane curve $C \subset \mathbb{P}^2$ as a surface-divisor pair $(\mathbb{P}^2, C)$ and define $\mathcal{M}_d$ as a moduli space of pairs (X, D) where X is a degeneration of the plane. We show that, if d is not divisible by 3, the stack $\mathcal{M}_d$ is smooth and the degenerate surfaces X can be described explicitly.
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1 Introduction

Let V_d be the moduli space of smooth plane curves of degree $d \geq 3$. Then V_d is the quotient $U_d / \operatorname{Aut}(\mathbb{P}^2)$ where U_d is the open locus of smooth curves in the Hilbert scheme H_d of plane curves of degree d. These moduli spaces are fundamental objects in algebraic geometry. Geometric invariant theory provides a compactification $\bar{V}_d$ of V_d. However, $\bar{V}_d$ is rather unsatisfactory for several reasons. First, $\bar{V}_d$ is not a moduli space itself — some points of the boundary correspond to several isomorphism classes of plane curves. Second, $\bar{V}_d$ has fairly complicated singularities at the boundary. In particular, these rule out the possibility of performing intersection theory on $\bar{V}_d$ to obtain enumerative results. Finally, the boundary is difficult to describe explicitly — there is a stratification given by the type of singularities on the degenerate curve, but this can only be computed for small degrees.

In this paper we describe an alternative compactification $\mathcal{M}_d$ of V_d. The space $\mathcal{M}_d$ is a moduli space of *stable pairs*. A stable pair is a surface-divisor pair (X, D) which is a degeneration of the plane together with a curve and satisfies certain additional properties. Morally speaking, the pair (X, D) should be identified with the curve D; the existence of an embedding $D \hookrightarrow X$ gives some structural information about D, e.g., the existence of a Brill–Noether special linear system on D. There is a stratification of $\mathcal{M}_d$

given by the isomorphism type of the surface X. If d is not divisible by 3 then we can explicitly describe the surfaces X which occur and so determine this stratification. Moreover, in this case, the space $\mathcal{M}_d$ is smooth (as a stack) and, writing $\mathcal{M}_d^0$ for the open stratum corresponding to the plane, the boundary $\mathcal{M}_d \backslash \mathcal{M}_d^0$ is a normal crossing divisor.

We pause to describe the simplest example, namely the case $d = 4$. The surfaces X occuring are the plane, the cone over the rational normal curve of degree 4 and the non-normal surface obtained by glueing two quadric cones along a ruling so that the vertices coincide. In the language of weighted projective spaces, the latter two surfaces are $\mathbb{P}(1, 1, 4)$ and $\mathbb{P}(1, 1, 2) \cup \mathbb{P}(1, 1, 2)$ respectively. The curves lying on $\mathbb{P}(1, 1, 4)$ are hyperelliptic — we obtain a 2-to-1 map to $\mathbb{P}^1$ by projecting away from the vertex. The curves lying on $\mathbb{P}(1, 1, 2) \cup \mathbb{P}(1, 1, 2)$ are 'degenerate hyperelliptic' — we obtain a 2-to-1 map to $\mathbb{P}^1 \cup \mathbb{P}^1$ by projecting away from the common vertex of the component surfaces; these curves have two components of genus 1 meeting in two nodes. The stratification of $\mathcal{M}_4$ is as follows: we have $\mathcal{M}_4 = Z_0 \cup Z_1 \cup Z_2$ where Z_0, Z_1 and Z_2 denote the strata corresponding to $\mathbb{P}^2$, $\mathbb{P}(1, 1, 4)$ and $\mathbb{P}(1, 1, 2) \cup \mathbb{P}(1, 1, 2)$. The stratum Z_0 is open, Z_1 is a locally closed locus of codimension 1, Z_2 is closed of codimension 2 and the closure of Z_1 is $Z_1 \cup Z_2$. The degree 4 case was originally treated by Hassett [Has], who worked with a different class of pairs (X, D). Roughly, we allow worse singularities on D in order to gain greater control of the surface X. The amazing thing is that, with our definition of stable pair, many of the features of the degree 4 case persist for all degrees which are not divisible by 3 (e.g. $\mathcal{M}_d$ is smooth and each degenerate surface X has at most two components).

We describe stable pairs in more detail. If (X, D) is a stable pair then the surface X has semi log canonical singularities (Definition 2.2) and the $\mathbb{Q}$-Cartier divisor $-K_X$ is ample. The divisor D lies in the linear system $|\frac{-d}{3}K_X|$ and has mild singularities. More carefully, the singularities of D which are permitted are precisely those such that the log canonical threshold of the pair (X, D) is strictly larger than $\frac{3}{d}$. For example, if $d = 4$, the singularities of D are either nodes or cusps.

There is a coarse classification of the surfaces X into types A, B, C and D. Type A are the normal surfaces. Type B have two normal components meeting in a smooth rational curve. Types C and D have several components forming an 'umbrella' or a 'fan' respectively. If the degree d is not divisible by 3 then only types A and B occur; in particular, X has at most 2 components. Moreover, the only singularities of X are quotients of smooth or normal crossing points.

We give an explicit description of the surfaces X of type A. If X is log terminal then X is obtained as a deformation of a weighted projective space $\mathbb{P}(a^2, b^2, c^2)$ where (a, b, c) is a solution of the Markov equation

$$a^2 + b^2 + c^2 = 3abc.$$

This is a refinement of a result of Manetti [Ma], so we call such surfaces *Manetti surfaces*. If X is not log terminal then X is an elliptic cone of degree 9. We also present a finer classification of the surfaces of type B.

We give a map of the paper. In Section 2 we define stable pairs and prove a completeness property, namely, that a family of smooth plane curves over a punctured curve can be completed to a family of stable pairs in a canonical way. In Section 3 we develop a theory of $\mathbb{Q}$-Gorenstein deformations for semi log canonical surfaces which we use to construct the moduli space of stable pairs $\mathcal{M}_d$ and study its infinitesimal properties. We construct the space $\mathcal{M}_d$ in Section 4. We also provide an effective bound on the index of a surface occurring in a stable pair in terms of the degree. In Section 5 we give the coarse classification of the degenerate surfaces X. In Section 6 we collect some restrictions on the singularities of X and the Picard numbers of the components implied by the existence of a smoothing of X to $\mathbb{P}^2$. Section 7 provides the simplifications in the case $3 \not| d$ stated above and Sections 8 and 9 give the classification of the type A and B surfaces respectively. In Section 10 we explain the relation between our notion of stability and GIT stability for a plane curve. Finally, in Section 11 we give the complete classification of stable pairs of degrees 4 and 5.

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2 Stable pairs

We define the notion of a stable pair and show that, possibly after base change, every family of smooth plane curves over a punctured curve can be completed to a family of stable pairs in a unique way. Equivalently, the moduli space of stable pairs is separated and proper. As a preliminary step, we define semistable pairs and show that every such family can be completed to a family of semistable pairs, although the completion is not necessarily uniquely determined.

We use the semistable minimal model program, which is explained in [KM], Chapter 7. Our construction is a refinement of the usual construction of compact moduli of pairs ([KSB],[Al]) applied to the case of pairs consisting of the plane together with a curve of degree d. The standard construction produces a moduli space $\mathcal{M}_d^\alpha$ of pairs (X, D) such that $(X, \alpha D)$ is semi log canonical (see Definition 2.2) and $K_X + \alpha D$ is ample for some fixed $\alpha \in \mathbb{Q}$; here we require $\alpha > \frac{3}{d}$ in order that $K_{\mathbb{P}^2} + \alpha D$ is ample for D a plane curve of degree d. However, there are technical problems in the construction of this moduli space, in particular, the correct definition of a family $(\mathcal{X}, \mathcal{D})/S$ of such pairs is unclear. The main problem is that we cannot insist that both the relative divisors $K_{\mathcal{X}}$ and $\mathcal{D}$ are $\mathbb{Q}$-Cartier. This complicates the deformation theory and thus renders an infinitesimal study of the moduli space intractable. We instead construct a moduli space $\mathcal{M}_d$ of *stable pairs*. A stable pair is a pair (X, D) such that $(X, (\frac{3}{d} + \epsilon)D)$ is semi log canonical and $K_X + (\frac{3}{d} + \epsilon)D$ is ample for all $0 < \epsilon \ll 1$. It is not immediately clear that stable pairs are bounded; however, once this is established, we deduce that ϵ may be chosen uniformly. That is, there exists $\epsilon_0 > 0$ such that for every stable pair (X, D), the pair $(X, (\frac{3}{d} + \epsilon)D)$ is semi log canonical and $K_X + (\frac{3}{d} + \epsilon)D$ is ample for any $0 < \epsilon \leq \epsilon_0$. Thus $\mathcal{M}_d^\alpha$ coincides with $\mathcal{M}_d$ for $\frac{3}{d} < \alpha \leq \frac{3}{d} + \epsilon_0$, or, more coarsely, $\mathcal{M}_d$ is the limit of $\mathcal{M}_d^\alpha$ as $\alpha \searrow \frac{3}{d}$. This was the original motivation for the definition of a stable pair. The space $\mathcal{M}_d$ is much easier to understand than the space $\mathcal{M}_d^\alpha$ for arbitrary α. Hence, in what follows, we construct $\mathcal{M}_d$ directly.

Notation 2.1. We always work over $\mathbb{C}$. We write $0 \in T$ for the germ of a smooth curve. We use script letters to denote flat families over T and regular letters for the special fibre, e.g.,

$$\begin{array}{ccc} X & \subset & \mathcal{X} \\ \downarrow & & \downarrow \\ 0 & \in & T \end{array}$$

We recall the definition of semi log canonical singularities of surface-divisor pairs ([KSB], [Al]). These are the singularities we must allow to compactify moduli of pairs.

Definition 2.2. Let X be a surface and D an effective $\mathbb{Q}$-divisor on X. The pair (X, D) is semi log canonical (respectively semi log terminal) if

(1) The surface X is Cohen-Macaulay and has only normal crossing singularities in codimension 1.

(2) Let K_X denote the Weil divisor class on X corresponding to the dualising sheaf ω_X. Then the divisor $K_X + D$ is $\mathbb{Q}$-Cartier.

(3) Let $\nu \colon X^\nu \to X$ be the normalisation of X. Let Δ denote the double curve of X and write D^ν and Δ^ν for the inverse images of D and Δ on X^ν. Then the pair $(X^\nu, \Delta^\nu + D^\nu)$ is log canonical (respectively log terminal).

We use the abbreviations slc and slt for semi log canonical and semi log terminal.

Remark 2.3. (1) The dualising sheaf ω_X satisfies Serre's condition S_2. It is also invertible in codimension 1 by (1). Hence it corresponds to a Weil divisor class K_X as stated. If X is normal this is of course the usual canonical divisor class.

(2) If (X, D) is slc then no component of D is contained in the double curve Δ by (3).

(3) Note that $K_{X^\nu} + \Delta^\nu + D^\nu = \nu^\star(K_X + D)$.

Definition 2.4. Let X be a surface and D an effective $\mathbb{Q}$-Cartier divisor on X. Let $d \in \mathbb{N}$, $d \geq 3$. The pair (X, D) is a *semistable pair* of degree d if

(1) The surface X is normal and log terminal.

(2) The pair $(X, \frac{3}{d}D)$ is log canonical.

(3) The divisor $dK_X + 3D$ is linearly equivalent to zero.

(4) There is a deformation $(\mathcal{X}, \mathcal{D})/T$ of the pair (X, D) over the germ of a curve such that the general fibre $\mathcal{X}_t$ of $\mathcal{X}/T$ is isomorphic to $\mathbb{P}^2$ and the divisors $K_{\mathcal{X}}$ and $\mathcal{D}$ are $\mathbb{Q}$-Cartier.

Remark 2.5. There is a very concrete classification of the surfaces X appearing here (Theorem 8.3).

Theorem 2.6. *Let $0 \in T$ be a germ of a curve and write $T^\times = T - 0$. Let $\mathcal{D}^\times \subset \mathbb{P}^2 \times T^\times$ be a family of smooth plane curves over $T^\times$ of degree $d \geq 3$. Then there exists a finite surjective base change $T' \to T$ and a family $(\mathcal{X}, \mathcal{D})/T'$ of semistable pairs extending the pullback of the family $(\mathbb{P}^2 \times T^\times, \mathcal{D}^\times)/T^\times$ such that the divisors $K_{\mathcal{X}}$ and $\mathcal{D}$ are $\mathbb{Q}$-Cartier.*

Proof. First complete $(\mathbb{P}^2 \times T^\times, \mathcal{D}^\times)$ to a flat family $(\mathbb{P}^2 \times T, \mathcal{D})$ over T. After a base change (which we will suppress in our notation) there is a semistable log resolution

$$\pi\colon (\tilde{\mathcal{X}}, \tilde{\mathcal{D}}) \to (\mathbb{P}^2 \times T, \mathcal{D})/T$$

which is an isomorphism over $T^\times$. We proceed as follows:

(1) Run a $K_{\tilde{\mathcal{X}}} + \frac{3}{d}\tilde{\mathcal{D}}$ MMP over T. Let $(\mathcal{X}_1, \mathcal{D}_1)/T$ denote the end product. Then $K_{\mathcal{X}_1} + \frac{3}{d}\mathcal{D}_1$ is relatively nef and vanishes on $\mathcal{X}_1^\times = \mathbb{P}^2 \times T^\times$; it follows that $dK_{\mathcal{X}_1} + 3\mathcal{D}_1 \sim 0$ by Lemma 2.7.

(2) Run a $K_{\mathcal{X}_1}$ MMP over T. The end product $(\mathcal{X}, \mathcal{D})/T$ is the required completion of $(\mathbb{P}^2 \times T^\times, \mathcal{D}^\times)$.

We verify the required properties of $(\mathcal{X}, \mathcal{D})/T$. We refer to [KM] Chapter 7 for background on the semistable minimal model program. The family $\mathcal{X}/T$ is a Mori fibre space since it is the end product of a MMP and the general fibre is a del Pezzo surface, namely $\mathbb{P}^2$. Regarding the singularities of $\mathcal{X}/T$, we know that the pair $(\mathcal{X}, X)$ is dlt and $\mathcal{X}$ is $\mathbb{Q}$-factorial. It follows that X is irreducible using $\rho(\mathcal{X}/T) = 1$ and $\mathbb{Q}$-factoriality. Then X is normal and log terminal by the dlt property.

The pair $(\mathcal{X}_1, X_1 + \frac{3}{d}\mathcal{D}_1)$ is dlt; since $dK_{\mathcal{X}_1} + 3\mathcal{D}_1 \sim 0$ it follows that $dK_{\mathcal{X}} + 3\mathcal{D} \sim 0$ and $(\mathcal{X}, X + \frac{3}{d}\mathcal{D})$ is log canonical. Thus $(X, \frac{3}{d}D)$ is log canonical and $dK_X + 3D \sim 0$ by adjunction. ☐

Lemma 2.7. *Let $\mathcal{X}/(0 \in T)$ be a flat family of projective slc surfaces over the germ of a curve such that the general fibre is normal. Let $\mathcal{X}^\times/T^\times$ denote the restriction of the family to the punctured curve $T^\times = T \backslash \{0\}$. Let $\mathcal{B}$ be a $\mathbb{Q}$-Cartier divisor on $\mathcal{X}$ such that $\mathcal{B}$ is relatively nef and $\mathcal{B}|_{\mathcal{X}^\times} \sim 0$. Then $\mathcal{B} \sim 0$.*

Proof. Let $X_1, \ldots, X_n$ denote the irreducible components of X, so $X = \sum X_i$ as divisors on $\mathcal{X}$. We have an exact sequence

$$0 \to \mathbb{Z}X \to \oplus \mathbb{Z}X_i \to \mathrm{Cl}(\mathcal{X}) \to \mathrm{Cl}(\mathcal{X}^\times) \to 0.$$

Hence, since $\mathcal{B}|_{\mathcal{X}^\times} \sim 0$, we may write $\mathcal{B} \sim \sum a_i X_i$, where $a_i \leq 0$ for all i and we have equality for some i. If $a_j = 0$, then $\mathcal{B}|_{X_j} = \sum_{i \neq j} a_i X_i|_{X_j} \leq 0$. But $\mathcal{B}|_{X_j}$ is nef, hence $\mathcal{B}|_{X_j} = 0$, i.e., $a_i = 0$ for each i such that X_i and X_j meet in a curve. It follows by induction that $a_i = 0$ for all i, i.e., $\mathcal{B} \sim 0$. ☐

Definition 2.8. Let X be a surface and D an effective $\mathbb{Q}$-Cartier divisor on X. Let $d \in \mathbb{N}$, $d \geq 4$. The pair (X, D) is a *stable pair* of degree d if

(1) The pair $(X, (\frac{3}{d}+\epsilon)D)$ is slc and the divisor $K_X + (\frac{3}{d}+\epsilon)D$ is ample for some $\epsilon > 0$.

(2) (=2.4(3)) The divisor $dK_X + 3D$ is linearly equivalent to zero.

(3) (=2.4(4)) There is a deformation $(\mathcal{X}, \mathcal{D})/T$ of the pair (X, D) over the germ of a curve such that the general fibre $\mathcal{X}_t$ of $\mathcal{X}/T$ is isomorphic to $\mathbb{P}^2$ and the divisors $K_{\mathcal{X}}$ and $\mathcal{D}$ are $\mathbb{Q}$-Cartier.

Remark 2.9. Conditions (1) and (2) may be replaced by the following (cf. our motivating remarks in the introduction of this section):

($1'$) The pair $(X, (\frac{3}{d}+\epsilon)D)$ is slc and the divisor $K_X + (\frac{3}{d}+\epsilon)D$ is ample for all $0 < \epsilon \ll 1$.

Clearly (1) and (2) imply ($1'$) and ($1'$) implies (1); it remains to show that ($1'$) (together with (3)) implies (2). If (X, D) satisfies ($1'$) then, since $K_X + (\frac{3}{d}+\epsilon)D$ is ample for all $0 < \epsilon \ll 1$, the limit $K_X + \frac{3}{d}D$ is nef. Suppose $(\mathcal{X}, \mathcal{D})/T$ is a smoothing of (X, D) as in (3). The divisor $dK_{\mathcal{X}} + 3\mathcal{D}$ is relatively nef and vanishes on the general fibre, hence is linearly equivalent to zero by Lemma 2.11(1) and Lemma 2.7. Thus $dK_X + 3D \sim 0$ by restriction, so (X, D) satisfies (2) as required.

Remark 2.10. We note that, if d is a multiple of 3, then $\frac{d}{3}K_X + D \sim 0$. For, writing $(\mathcal{X}, \mathcal{D})/T$ for a smoothing as above, the condition $dK_X + 3D \sim 0$ implies that $dK_{\mathcal{X}} + 3\mathcal{D} \sim 0$ and $\mathrm{Cl}(\mathcal{X})$ is torsion-free by Lemma 2.11, hence $\frac{d}{3}K_{\mathcal{X}} + \mathcal{D} \sim 0$ and so $\frac{d}{3}K_X + D \sim 0$ by restriction.

Lemma 2.11. *Let $\mathcal{X}/(0 \in T)$ be a flat family of surfaces over the germ of a curve with general fibre $\mathbb{P}^2$ and reduced special fibre X. Then*

(1) $\mathcal{X}^\times \cong \mathbb{P}^2 \times T^\times$

(2) $\mathrm{Cl}(\mathcal{X}) \cong \mathbb{Z}^n$*, where n is the number of components of X.*

Proof. Since the general fibre is $\mathbb{P}^2$ there is no monodromy and $\mathcal{X}^\times \cong \mathbb{P}^2 \times T^\times$. Hence $\mathrm{Cl}(\mathcal{X}^\times) \cong \mathbb{Z}$. The exact sequence

$$0 \to \mathbb{Z}X \to \oplus \mathbb{Z}X_i \to \mathrm{Cl}(\mathcal{X}) \to \mathrm{Cl}(\mathcal{X}^\times) \to 0$$

now gives $\mathrm{Cl}(\mathcal{X}) \cong \mathbb{Z}^n$ as claimed. □

Theorem 2.12. *Let $\mathcal{D}^\times \subset \mathbb{P}^2 \times T^\times$ be a family of smooth plane curves of degree $d \geq 4$ over a punctured curve $T^\times$. Then there exists a finite surjective base change $T' \to T$ and a family $(\mathcal{X}, \mathcal{D})/T'$ of stable pairs extending the* *pullback of the family* $(\mathbb{P}^2 \times T^\times, \mathcal{D}^\times)/T^\times$ *such that the divisors* $K_\mathcal{X}$ *and* $\mathcal{D}$ *are* $\mathbb{Q}$*-Cartier. Moreover the family* $(\mathcal{X}, \mathcal{D})/T'$ *is unique in the following sense: any two such families become isomorphic after a further finite surjective base change.*

Proof. Let $(\mathcal{X}_1, \mathcal{D}_1)/T$ be a family of semistable pairs extending the family $(\mathbb{P}^2 \times T^\times, \mathcal{D}^\times)/T^\times$ as constructed in the proof of Theorem 2.6. Then the pair $(\mathcal{X}_1, X_1 + \frac{3}{d}\mathcal{D}_1)$ is log canonical and the pair $(\mathcal{X}_1, X_1)$ is dlt. There exists a partial semistable resolution (a 'maximal crepant blowup' of $(\mathcal{X}_1, \frac{3}{d}\mathcal{D}_1)$)

$$\pi \colon (\mathcal{X}_2, \mathcal{D}_2) \to (\mathcal{X}_1, \mathcal{D}_1)/T$$

such that $dK_{\mathcal{X}_2} + 3\mathcal{D}_2 = \pi^\star(dK_{\mathcal{X}_1} + 3\mathcal{D}_1) \sim 0$ and $(\mathcal{X}_2, X_2 + (\frac{3}{d} + \epsilon)\mathcal{D}_2)$ is dlt for $0 < \epsilon \ll 1$. Let $(\mathcal{X}, \mathcal{D})/T$ be the $K_{\mathcal{X}_2} + (\frac{3}{d} + \epsilon)\mathcal{D}_2$ canonical model. Then $(\mathcal{X}, X + (\frac{3}{d} + \epsilon)\mathcal{D})$ is log canonical, the divisor $dK_\mathcal{X} + 3\mathcal{D} \sim 0$ and $K_\mathcal{X} + X + (\frac{3}{d} + \epsilon)\mathcal{D}$ is relatively ample. By adjunction $(X, (\frac{3}{d} + \epsilon)D)$ is slc, the divisor $dK_X + 3D \sim 0$ and $K_X + (\frac{3}{d} + \epsilon)D$ is ample. Note also that $K_\mathcal{X} + (\frac{3}{d} + \epsilon)\mathcal{D}$ is $\mathbb{Q}$-Cartier by construction. Hence $K_\mathcal{X}$ and $\mathcal{D}$ are $\mathbb{Q}$-Cartier since $dK_\mathcal{X} + 3\mathcal{D} \sim 0$.

To prove uniqueness, note that $(\mathcal{X}, \mathcal{D})/T$ is the $K_{\tilde{\mathcal{X}}} + (\frac{3}{d} + \epsilon)\tilde{\mathcal{D}}$ canonical model of any semistable log resolution $(\tilde{\mathcal{X}}, \tilde{\mathcal{D}})/T$, where $\epsilon > 0$ is sufficiently small. □

We record the following important result, which is an immediate consequence of conditions (1) and (2) of Definition 2.8.

Proposition 2.13. *Let* (X, D) *be a stable pair. Then* X *is an slc surface and the divisor* $-K_X$ *is ample.*

3 $\mathbb{Q}$-Gorenstein deformation theory

We define the $\mathbb{Q}$-Gorenstein deformations of a slc surface X to be those locally induced by a deformation of the canonical covering of X. We then describe how to calculate the $\mathbb{Q}$-Gorenstein deformations of a given surface X. This theory is used in Section 4 to construct the moduli space $\mathcal{M}_d$ of stable pairs and in Section 7 to prove that $\mathcal{M}_d$ is smooth if $3 \not| d$. It can also be used to construct compact moduli spaces of surfaces of general type with a finer scheme theoretic structure than that originally defined in [KSB] and facilitates an infinitesimal study of such moduli spaces. My presentation here is influenced by earlier work of Kollár and Hassett [Has].

If a sheaf $\mathcal{F}$ on a surface X satisfies the S_2 condition, one can recover $\mathcal{F}$ from $\mathcal{F}|_U$ where $U \hookrightarrow X$ has finite complement. We require a relative S_2 condition for sheaves on families of slc surfaces which allows us to do this in the relative context. The definition and basic results are collected in Appendix A.

3.1 Definition of $\mathbb{Q}$-Gorenstein deformations

Let $P \in X$ be an slc surface germ. We define the *canonical covering* $\pi\colon Z \to X$ by

$$Z = \underline{\mathrm{Spec}}_X(\mathcal{O}_X \oplus \mathcal{O}_X(K_X) \oplus \cdots \oplus \mathcal{O}_X((N-1)K_X)),$$

where N is the index of $P \in X$ and the multiplication is given by fixing an isomorphism $\mathcal{O}_X(NK_X) \xrightarrow{\sim} \mathcal{O}_X$. This is a straightforward generalisation of the usual construction for X a normal variety such that K_X is $\mathbb{Q}$-Cartier (cf. [YPG]). It is characterised by the following properties:

(1) The morphism π is a cyclic quotient of degree N which is étale in codimension 1.

(2) The surface Z is Gorenstein, i.e., it is Cohen-Macaulay and the Weil divisor K_Z is Cartier.

For X an slc surface, the canonical covering at a point $P \in X$ is uniquely determined in the étale topology. Hence the data of canonical coverings everywhere locally on X defines a Deligne-Mumford stack $\mathfrak{X}$ with coarse moduli space X, the *canonical covering stack* of X (cf. [Ka], p. 18, Definition 6.1).

Definition 3.1. Let $P \in X$ be an slc surface germ. Let N be the index of X and $Z \to X$ the canonical covering, a μ_N quotient. We say a deformation $\mathcal{X}/(0 \in S)$ of X is $\mathbb{Q}$-*Gorenstein* if there is a μ_N-equivariant deformation $\mathcal{Z}/S$ of Z whose quotient is $\mathcal{X}/S$.

Notation 3.2. Let $\mathcal{X}/S$ be a flat family of slc surfaces. Let $i\colon \mathcal{X}^0 \hookrightarrow \mathcal{X}/S$ be the inclusion of the Gorenstein locus of $\mathcal{X}/S$, i.e., the locus where the relative dualising sheaf $\omega_{\mathcal{X}/S}$ is invertible. We write $\omega_{\mathcal{X}/S}^{[N]}$ for the sheaf $i_\star\omega_{\mathcal{X}^0/S}^{\otimes N}$.

We say that a family $\mathcal{X}/S$ is *weakly* $\mathbb{Q}$-*Gorenstein* if the sheaf $\omega_{\mathcal{X}/S}^{[N]}$ is invertible for some $N \geq 1$ (cf. [KSB]). The least such N is the *index* of $\mathcal{X}/S$. If $\mathcal{X}$ is normal and S is smooth this is just the requirement that $K_{\mathcal{X}}$ is $\mathbb{Q}$-Cartier. We show that a $\mathbb{Q}$-Gorenstein family is weakly $\mathbb{Q}$-Gorenstein

(Lemma 3.3). Moreover, if the base S is a curve and the general fibre of $\mathcal{X}/S$ is canonical then the two conditions are equivalent (Lemma 3.4).

Lemma 3.3. *Let $P \in X$ be an slc surface germ of index N. Let $\mathcal{X}/(0 \in S)$ be a $\mathbb{Q}$-Gorenstein deformation of X. Then $\mathcal{X}/S$ is weakly $\mathbb{Q}$-Gorenstein of index N.*

Proof. There is a diagram

$$\begin{array}{ccc} Z & \subset & \mathcal{Z} \\ \downarrow & & \downarrow \\ X & \subset & \mathcal{X} \\ \downarrow & & \downarrow \\ 0 & \in & S \end{array}$$

where Z is the canonical cover of $P \in X$ and $\mathcal{Z}/S$ is a μ_N-equivariant deformation of Z with quotient $\mathcal{X}/S$. We have an isomorphism

$$\omega_{\mathcal{Z}/S} \otimes k(0) \cong \omega_Z \cong \mathcal{O}_Z$$

by the base change property for the relative dualising sheaf. Hence $\omega_{\mathcal{Z}/S} \cong \mathcal{O}_{\mathcal{Z}}$ by Nakayama's lemma applied to the $\mathcal{O}_{\mathcal{Z}}$-module $\omega_{\mathcal{Z}/S}$. Thus $\omega_{\mathcal{Z}/S}^{\otimes N}$ is invertible and has a μ_N-invariant generator. Now, let $i\colon \mathcal{X}^0 \hookrightarrow \mathcal{X}$ denote the Gorenstein locus of $\mathcal{X}/S$ and $\pi^0\colon \mathcal{Z}^0 \to \mathcal{X}^0$ the restriction of the covering $\pi\colon \mathcal{Z} \to \mathcal{X}$. Then π^0 is an étale μ_N quotient, hence

$$\omega_{\mathcal{X}^0/S}^{\otimes N} \cong (\pi^0_\star \omega_{\mathcal{Z}^0/S}^{\otimes N})^{\mu_N} \cong (\pi^0_\star \mathcal{O}_{\mathcal{Z}^0})^{\mu_N} \cong \mathcal{O}_{\mathcal{X}^0}.$$

Applying $i_\star$ we obtain $\omega_{\mathcal{X}/S}^{[N]} \cong \mathcal{O}_{\mathcal{X}}$, thus $\mathcal{X}/S$ is weakly $\mathbb{Q}$-Gorenstein. To prove that N is the index, suppose $\omega_{\mathcal{X}/S}^{[M]}$ is invertible for some $M \in \mathbb{N}$, and consider the natural map

$$\omega_{\mathcal{X}/S}^{[M]} \otimes k(0) \to \omega_X^{[M]}.$$

The map is an isomorphism in codimension 1, and both sheaves are S_2, hence it is an isomorphism. So $\omega_X^{[M]}$ is invertible and N divides M. $\square$

Lemma 3.4. *Let $\mathcal{X}/(0 \in T)$ be a flat family of slc surfaces over the germ of a curve. Suppose that the general fibre is canonical, i.e., has only Du Val singularities, and that $K_{\mathcal{X}}$ is $\mathbb{Q}$-Cartier. Then $\mathcal{X}/T$ is $\mathbb{Q}$-Gorenstein.*

Proof. We work locally at a point $P \in \mathcal{X}$. Let $Z \to X$ and $\mathcal{Z} \to \mathcal{X}$ be the canonical covering of $P \in X$ and $P \in \mathcal{X}$ respectively. Note that the index of X equals the index of $\mathcal{X}$ ([KSB], Lemma 3.16, p. 316), hence these maps have the same degree. We need to show that $\mathcal{Z}/T$ is a deformation of Z. Since the fibre $\mathcal{Z}_0$ agrees with Z over $X - P$, it is enough to show that $\mathcal{Z}_0$ is Cohen-Macaulay. The fibre X of $\mathcal{X}/T$ is slc, so the pair $(\mathcal{X}, X)$ is log canonical — this is an 'inversion of adjunction' type result. In more detail, after a finite surjective base change $T' \to T$, there is a semistable resolution $\pi \colon \tilde{\mathcal{X}}' \to \mathcal{X}' = \mathcal{X} \times_T T'$. Then the proof of [KSB], Theorem 5.1(a) shows that $\mathcal{X}'/T'$ coincides with the canonical model of $\tilde{\mathcal{X}}'$ over $\mathcal{X}'$. Hence $(\mathcal{X}', X')/T'$ is log canonical. Finally, writing $g \colon \mathcal{X}' \to \mathcal{X}$ for the map induced by the base change $T' \to T$, we have $K_{\mathcal{X}'} + X' = g^\star(K_{\mathcal{X}} + X)$ by Riemann-Hurwitz, so $(\mathcal{X}, X)$ is log canonical by [KM], Proposition 5.20(4). Since X is Cartier and the general fibre is canonical it follows that $\mathcal{X}$ is canonical. Hence the cover $\mathcal{Z}$ is also canonical, so in particular Cohen-Macaulay. Then the fibre $\mathcal{Z}_0 = (t = 0) \subset \mathcal{Z}$ is also Cohen-Macaulay. $\square$

3.2 Computing $\mathbb{Q}$-Gorenstein deformations

For $\mathcal{X}/S$ a $\mathbb{Q}$-Gorenstein family of slc surfaces, we define the canonical covering stack $\mathfrak{X}/S$ of the family $\mathcal{X}/S$, and show that the infinitesimal $\mathbb{Q}$-Gorenstein deformations of $\mathcal{X}/S$ correspond exactly to the infinitesimal deformations of $\mathfrak{X}/S$ (defined carefully below). We can then apply the results of [I1],[I2] to compute the $\mathbb{Q}$-Gorenstein deformations of $\mathcal{X}/S$ (Theorem 3.9). Note that, for our explicit computations in Sections 8 and 9, we need only consider infinitesimal $\mathbb{Q}$-Gorenstein deformations of an slc surface $X/\mathbb{C}$. However, we must develop the theory for $\mathbb{Q}$-Gorenstein families over an arbitrary affine scheme in order to establish 'openness of versality' for $\mathbb{Q}$-Gorenstein deformations (cf. [Ar], Section 4). This is used in the construction of the moduli space of stable pairs in Section 4.

The following lemma motivates the definition of the canonical covering stack of a $\mathbb{Q}$-Gorenstein family.

Lemma 3.5. *Let $P \in X$ be an slc surface germ of index N and $Z \to X$ the canonical covering with group $G \cong \mu_N$. Let $\mathcal{Z}/(0 \in S)$ be a G-equivariant deformation of Z inducing a $\mathbb{Q}$-Gorenstein deformation $\mathcal{X}/(0 \in S)$ of X. Then there is an isomorphism*

$$\mathcal{Z} \cong \underline{\mathrm{Spec}}_{\mathcal{X}}(\mathcal{O}_{\mathcal{X}} \oplus \omega_{\mathcal{X}/S} \oplus \cdots \oplus \omega_{\mathcal{X}/S}^{[N-1]})$$

where the multiplication is given by fixing a trivialisation of $\omega_{\mathcal{X}/S}^{[N]}$. In particular, $\mathcal{Z}/S$ is determined by $\mathcal{X}/S$.

Proof. Let $i\colon \mathcal{X}^0 \hookrightarrow \mathcal{X}$ denote the open locus where the covering $\pi\colon \mathcal{Z} \to \mathcal{X}$ is étale and let $\pi^0\colon \mathcal{Z}^0 \to \mathcal{X}^0$ denote the restriction of the covering. The map π^0 is an étale μ_N quotient, hence

$$\mathcal{Z}^0 \cong \underline{\mathrm{Spec}}_{\mathcal{X}^0}(\mathcal{O}_{\mathcal{X}^0} \oplus \mathcal{L} \oplus \cdots \oplus \mathcal{L}^{\otimes N-1})$$

for some line bundle $\mathcal{L}$ on $\mathcal{X}^0$, with multiplication given by an isomorphism $\mathcal{L}^{\otimes N} \cong \mathcal{O}_{\mathcal{X}}$. Here the sheaves $\mathcal{L}^{\otimes r}$ are the eigensheaves of the G action on $\pi^0_\star \mathcal{O}_{\mathcal{Z}^0}$. Since $\mathcal{Z}$ is a deformation of the canonical covering of Z of X, we may assume that the restriction of $\mathcal{L}$ to the fibre X^0 is identified with ω_{X^0}. Now $\omega_{\mathcal{X}^0/S} = (\pi^0_\star \omega_{\mathcal{Z}^0/S})^G$ and $\omega_{\mathcal{Z}/S} \cong \mathcal{O}_{\mathcal{Z}}$, hence $\omega_{\mathcal{X}^0/S}$ is isomorphic to a G-eigensheaf of $\pi^0_\star \mathcal{O}_{\mathcal{Z}^0}$ and so $\omega_{\mathcal{X}^0/S} \cong \mathcal{L}$ by our choice of $\mathcal{L}$. Finally, $\mathcal{Z}$ is determined by its restriction $\mathcal{Z}^0$ since $\mathcal{Z}$ is S_2 over S, so we obtain an isomorphism as claimed. ☐

Let $\mathcal{X}/S$ be a $\mathbb{Q}$-Gorenstein family of slc surfaces. For $P \in \mathcal{X}/S$ a point of index N, we define the *canonical covering* $\pi\colon \mathcal{Z} \to \mathcal{X}$ of $P \in \mathcal{X}/S$ by

$$\mathcal{Z} = \underline{\mathrm{Spec}}_{\mathcal{X}}(\mathcal{O}_{\mathcal{X}} \oplus \omega_{\mathcal{X}/S} \oplus \cdots \oplus \omega_{\mathcal{X}/S}^{[N-1]}),$$

where the multiplication is given by fixing a trivialisation of $\omega_{\mathcal{X}/S}^{[N]}$ at P. The canonical covering of $P \in \mathcal{X}/S$ is uniquely determined in the étale topology. Hence the data of canonical coverings everywhere locally on $\mathcal{X}/S$ defines a Deligne-Mumford stack $\mathfrak{X}/S$ with coarse moduli space $\mathcal{X}/S$, the *canonical covering stack* of $\mathcal{X}/S$.

The stack $\mathfrak{X}/S$ is flat over S by Lemma 3.5. Moreover, for any base change $T \to S$, let $\mathfrak{X}_T$ denote the canonical covering stack of $\mathcal{X} \times_S T/T$, then there is a canonical isomorphism $\mathfrak{X}_T \xrightarrow{\sim} \mathfrak{X} \times_S T$. For, given an étale neighbourhood $\mathcal{Z} \to \mathfrak{X}$ as above, there is a corresponding étale neighbourhood $\mathcal{Z}_T \to \mathfrak{X}_T$ and a natural map $\mathcal{Z}_T \to \mathcal{Z} \times_S T$ by the base change property for $\omega_{\mathcal{X}/S}$. The map is an isomorphism over the Gorenstein locus of $\mathcal{X} \times_S T/T$ and both $\mathcal{Z}_T$ and $\mathcal{Z} \times_S T$ are S_2 over T by Lemma 3.5, hence it is an isomorphism.

We collect some easy properties of the canonical covering stack $\mathfrak{X}/S$. There is a notion of an étale map $U \to \mathfrak{X}$ and hence the notion of sheaves on the étale site $\mathfrak{X}_{et}$ of the stack $\mathfrak{X}$. We shall only consider sheaves on $\mathfrak{X}_{et}$, and refer simply to 'sheaves on $\mathfrak{X}$'. Let $\pi\colon \mathcal{Z} \to \mathcal{X}$ be a local canonical covering at $P \in \mathcal{X}/S$, with group $G \cong \mu_N$. Then $\mathfrak{X}$ has local patch $[\mathcal{Z}/G]$

over $P \in \mathcal{X}$. Sheaves on $[\mathcal{Z}/G]$ correspond to G-equivariant sheaves on $\mathcal{Z}$. Let $p\colon \mathfrak{X} \to \mathcal{X}$ be the induced map to the coarse moduli space. Thus, locally, p is the map $[\mathcal{Z}/G] \to \mathcal{Z}/G$. If $\mathcal{F}$ is a sheaf on $[\mathcal{Z}/G]$ and $\mathcal{F}_{\mathcal{Z}}$ is the corresponding G-equivariant sheaf on $\mathcal{Z}$, then $p_\star\mathcal{F} = (\pi_\star\mathcal{F}_{\mathcal{Z}})^G$. In particular, the functor $p_\star$ is exact. For, the map π is finite and $(\pi_\star\mathcal{F}_{\mathcal{Z}})^G$ is a direct summand of $\pi_\star\mathcal{F}_{\mathcal{Z}}$ since we are in characteristic zero.

Let A be a $\mathbb{C}$-algebra and $A' \to A$ an infinitesimal extension. Let $\mathcal{X}/A$ be a $\mathbb{Q}$-Gorenstein family of slc surfaces and $\mathfrak{X}/A$ the canonical covering stack of $\mathcal{X}/A$. A *deformation* of $\mathfrak{X}/A$ over A' is a Deligne-Mumford stack $\mathfrak{X}'/A'$, flat over A', together with an isomorphism $\mathfrak{X}' \times_{\operatorname{Spec} A'} \operatorname{Spec} A \cong \mathfrak{X}$. Observe that, since the extension $A' \to A$ is infinitesimal, we may identify the étale sites of $\mathfrak{X}'$ and $\mathfrak{X}$. Thus, equivalently, a deformation $\mathfrak{X}'/A'$ of $\mathfrak{X}/A$ is a sheaf $\mathcal{O}_{\mathfrak{X}'}$ of flat A'-algebras on the étale site of $\mathfrak{X}$, together with an isomorphism $\mathcal{O}_{\mathfrak{X}'} \otimes_{A'} A \cong \mathcal{O}_{\mathfrak{X}}$. From this point of view, infinitesimal deformations of stacks fit into the general framework of [I1],[I2]. The stack $\mathfrak{X}/A$ is identified with the 'ringed topos' over A given by the étale site of $\mathfrak{X}$ together with the structure sheaf $\mathcal{O}_{\mathfrak{X}}$. The *cotangent complex* $L_{\mathfrak{X}/A}$ of $\mathfrak{X}/A$ is a complex of $\mathcal{O}_{\mathfrak{X}}$-modules L^i in degrees $i \le 0$, with $H^0(L_{\mathfrak{X}/A}) = \Omega_{\mathfrak{X}/A}$. For an extension $A' \to A$ whose kernel M satisfies $M^2 = 0$, the groups $\operatorname{Ext}^i(L_{\mathfrak{X}/A}, \mathcal{O}_{\mathfrak{X}} \otimes_A M)$, $i = 0, 1, 2$, control the deformations of $\mathfrak{X}/A$ over A'. We refer to [I2], Section 1 for a review of cotangent complex theory, and to [I1] for the definitive treatment.

In our calculations, we shall require the local-to-global spectral sequence for Ext and the Leray spectral sequence for stacks. These are derived for ringed topoi, and thus for stacks, in [SGA4], Exposé V. In particular, if $\mathfrak{X}/A$ is the canonical covering stack of a $\mathbb{Q}$-Gorenstein family $\mathcal{X}/A$ and $p\colon \mathfrak{X} \to \mathcal{X}$ the induced map, then $H^i(\mathfrak{X}, \mathcal{F}) = H^i(\mathcal{X}, p_\star\mathcal{F})$ for $\mathcal{F}$ a sheaf on $\mathfrak{X}$, since $p_\star$ is exact.

Notation 3.6. Let A be a $\mathbb{C}$-algebra and M a finite A-module. For $\mathcal{X}/A$ a flat family of schemes over A, let $L_{\mathcal{X}/A}$ denote the cotangent complex of $\mathcal{X}/A$. Define

$$T^i(\mathcal{X}/A, M) = \operatorname{Ext}^i(L_{\mathcal{X}/A}, \mathcal{O}_{\mathcal{X}} \otimes_A M)$$
$$\mathcal{T}^i(\mathcal{X}/A, M) = \mathcal{E}xt^i(L_{\mathcal{X}/A}, \mathcal{O}_{\mathcal{X}} \otimes_A M)$$

For $\mathcal{X}/A$ a $\mathbb{Q}$-Gorenstein family of slc surfaces over A, let $\mathfrak{X}/A$ denote the canonical covering stack of $\mathcal{X}/A$ and $p\colon \mathfrak{X} \to \mathcal{X}$ the induced map. Define

$$T^i_{QG}(\mathcal{X}/A, M) = \operatorname{Ext}^i(L_{\mathfrak{X}/A}, \mathcal{O}_{\mathfrak{X}} \otimes_A M)$$
$$\mathcal{T}^i_{QG}(\mathcal{X}/A, M) = p_\star\,\mathcal{E}xt^i(L_{\mathfrak{X}/A}, \mathcal{O}_{\mathfrak{X}} \otimes_A M)$$

Proposition 3.7. *Let $\mathcal{X}/A$ be a $\mathbb{Q}$-Gorenstein family of slc surfaces and $\mathfrak{X}/A$ the canonical covering stack. Let $A' \to A$ be an infinitesimal extension of A'. For $\mathcal{X}'/A'$ a $\mathbb{Q}$-Gorenstein deformation of $\mathcal{X}/A$, let $\mathfrak{X}'/A'$ denote the canonical covering stack of $\mathcal{X}'/A'$. Then the map $\mathcal{X}'/A' \mapsto \mathfrak{X}'/A'$ gives a bijection between the set of isomorphism classes of $\mathbb{Q}$-Gorenstein deformations of $\mathcal{X}/A$ over A' and the set of isomorphism classes of deformations of $\mathfrak{X}/A$ over A'.*

Proof. If $\mathcal{X}'/A'$ is a $\mathbb{Q}$-Gorenstein deformation of $\mathcal{X}/A$ then the canonical covering stack $\mathfrak{X}'/A'$ is a deformation of $\mathfrak{X}/A$. Conversely, if $\mathfrak{X}'/A'$ is a deformation of $\mathfrak{X}/A$ then the coarse moduli space $\mathcal{X}'/A'$ is a $\mathbb{Q}$-Gorenstein deformation of $\mathcal{X}/A$. It only remains to prove that, if $\mathfrak{X}'/A'$ is a deformation of $\mathfrak{X}/A$ with coarse moduli space $\mathcal{X}'/A'$, then the canonical covering stack $\tilde{\mathfrak{X}}'/A'$ of $\mathcal{X}'/A'$ is isomorphic to $\mathfrak{X}'/A'$. By induction, we may assume that the kernel M of $A' \to A$ satisfies $M^2 = 0$. Then the deformations of $\mathfrak{X}/A$ over A' form an affine space under $T^1_{QG}(\mathcal{X}/A, M)$ by [I2], Theorem 1.7. Let $\mathfrak{X}'/A'$ and $\tilde{\mathfrak{X}}'/A'$ differ by an element $t \in T^1_{QG}(\mathcal{X}/A, M)$; we show that $t = 0$. We have an exact sequence

$$0 \to H^1(\mathcal{T}^0_{QG}(\mathcal{X}/A, M)) \to T^1_{QG}(\mathcal{X}/A, M) \xrightarrow{\theta} H^0(\mathcal{T}^1_{QG}(\mathcal{X}/A, M))$$

obtained from the local-to-global spectral sequence for Ext on the stack $\mathfrak{X}$. The deformations $\tilde{\mathfrak{X}}'/A'$ and $\mathfrak{X}'/A'$ of $\mathfrak{X}/A$ induce isomorphic deformations locally by Lemma 3.5, hence $\theta(t) = 0$, i.e., $t \in H^1(\mathcal{T}^0_{QG}(\mathcal{X}/A, M))$. The natural map $\mathcal{T}^0_{QG}(\mathcal{X}/A, M) \to \mathcal{T}^0(\mathcal{X}/A, M)$ is an isomorphism by Lemma 3.8, so t is identified with the element of $H^1(\mathcal{T}^0(\mathcal{X}/A, M))$ relating the deformations of $\mathcal{X}/A$ induced by $\mathfrak{X}'/A'$ and $\tilde{\mathfrak{X}}'/A'$. But these deformations coincide by assumption, hence $t = 0$ as required. □

Lemma 3.8. *Let $\mathcal{X}/A$ be a $\mathbb{Q}$-Gorenstein family of slc surfaces and M a finite A-module. Then the natural map $\mathcal{T}^0_{QG}(\mathcal{X}/A, M) \to \mathcal{T}^0(\mathcal{X}/A, M)$ is an isomorphism.*

Proof. We work locally at $P \in \mathcal{X}$. Let $\pi \colon \mathcal{Z} \to \mathcal{X}$ be the canonical covering of $\mathcal{X}/A$, with covering group G, and $\mathfrak{X} = [\mathcal{Z}/G]$ the canonical covering stack. Then $\mathcal{T}^0_{QG}(\mathcal{X}/A, M) = (\pi_\star \mathcal{T}^0(\mathcal{Z}/A, M))^G$. The natural map $\mathcal{T}^0_{QG}(\mathcal{X}/A, M) \to \mathcal{T}^0(\mathcal{X}/A, M)$ is an isomorphism over the locus where the covering π is étale, hence it suffices to show that $\mathcal{T}^0_{QG}(\mathcal{X}/A, M)$ and $\mathcal{T}^0(\mathcal{X}/A, M)$ are weakly S_2 over A. First, we have

$$\mathcal{T}^0(\mathcal{X}/A, M) = \mathcal{H}om(L_{\mathcal{X}/A}, \mathcal{O}_\mathcal{X} \otimes_A M) = \mathcal{H}om(\Omega_{\mathcal{X}/A}, \mathcal{O}_\mathcal{X} \otimes_A M)$$

since the complex $L_{\mathcal{X}/A}$ has cohomology $\Omega_{\mathcal{X}/A}$ in degree 0. We claim that $\mathcal{O}_{\mathcal{X}} \otimes_A M$ is weakly S_2 over A, then $\mathcal{T}^0(\mathcal{X}/A, M)$ is weakly S_2 over A by Lemma A.5(1). To prove the claim, we may assume that $M = A/p$ for some prime ideal $p \subset A$ by A.5(2). In this case $\mathcal{O}_{\mathcal{X}} \otimes_A M$ is S_2 over A/p and so weakly S_2 over A as desired. Second, the sheaf $\mathcal{T}^0(\mathcal{Z}/A, M)$ is weakly S_2 over A as above, so $\pi_\star \mathcal{T}^0(\mathcal{Z}/A, M)$ is weakly S_2 over A. Since $(\pi_\star \mathcal{T}^0(\mathcal{Z}/A, M))^G$ is a direct summand of $\pi_\star \mathcal{T}^0(\mathcal{Z}/A, M)$, it is also weakly S_2 over A. ☐

Theorem 3.9. *Let $\mathcal{X}_0/A_0$ be a $\mathbb{Q}$-Gorenstein family of slc surfaces. Let M be a finite A_0-module.*

1. *The set of isomorphism classes of $\mathbb{Q}$-Gorenstein deformations of $\mathcal{X}_0/A_0$ over $A_0 + M$ is naturally an A_0-module and is canonically isomorphic to $T^1_{QG}(\mathcal{X}_0/A_0, M)$. Here $A_0 + M$ denotes the ring $A_0[M]$, with $M^2 = 0$.*
2. *Let $A \to A_0$ be an infinitesimal extension and $A' \to A$ a further extension with kernel the A_0-module M. Let $\mathcal{X}/A$ be a $\mathbb{Q}$-Gorenstein deformation of $\mathcal{X}_0/A_0$.*
   (a) *There is a canonical element $o(\mathcal{X}/A, A') \in T^2_{QG}(\mathcal{X}_0/A_0, M)$ which vanishes if and only if there exists a $\mathbb{Q}$-Gorenstein deformation $\mathcal{X}'/A'$ of $\mathcal{X}/A$ over A'.*
   (b) *If $o(\mathcal{X}/A, A') = 0$, the set of isomorphism classes of $\mathbb{Q}$-Gorenstein deformations $\mathcal{X}'/A'$ is an affine space under $T^1_{QG}(\mathcal{X}_0/A_0, M)$.*

Proof. The $\mathbb{Q}$-Gorenstein deformations of $\mathcal{X}_0/A_0$ are identified with the deformations of the canonical covering stack $\mathfrak{X}_0/A_0$ of $\mathcal{X}_0/A_0$ by Proposition 3.7. Hence the theorem follows from [I2], Theorem 1.5.1 and Theorem 1.7. Note that, in part (2), we have used the natural isomorphisms $T^i_{QG}(\mathcal{X}_0/A_0, M) \xrightarrow{\sim} T^i_{QG}(\mathcal{X}/A, M)$ given by [I2], 1.3. ☐

As remarked earlier, we need only consider infinitesimal deformations of an slc surface $X/\mathbb{C}$ for our later explicit computations. In the notation of the theorem, we may assume that $A_0 = \mathbb{C}$ and $M \cong \mathbb{C}$. We collect some useful notation and facts in this case below. Define $T^i_X, \mathcal{T}^i_X, T^i_{QG,X}, \mathcal{T}^i_{QG,X}$ by $T^i_X = T^i(X/\mathbb{C}, \mathbb{C})$ etc. By the Theorem, first order $\mathbb{Q}$-Gorenstein deformations of $X/\mathbb{C}$ are identified with $T^1_{QG,X}$ and the obstructions to extending $\mathbb{Q}$-Gorenstein deformations lie in $T^2_{QG,X}$. We have $\mathcal{T}^0_{QG,X} = \mathcal{T}^0_X = \mathcal{H}om(\Omega_X, \mathcal{O}_X)$, the tangent sheaf of X, by Lemma 3.8. Working locally at $P \in X$, let $\pi\colon Z \to X$ be the canonical covering, with group G, then

$\mathcal{T}^i_{QG,X} = (\pi_\star \mathcal{T}^i_Z)^G$. The sheaf $\mathcal{T}^1_Z$ is supported on the singular locus of Z and $\mathcal{T}^2_Z$ is supported on the locus where Z is not a local complete intersection. Finally, there is a local-to-global spectral sequence

$$E_2^{pq} = H^p(\mathcal{T}^q_{QG,X}) \Rightarrow T^{p+q}_{QG,X}$$

given by the local-to-global spectral sequence for Ext on the canonical covering stack of X.

3.3 Deformations of pairs

Finally, we study deformations of stable pairs (X, D). We prove that the presence of the divisor D does not produce any further obstructions.

Definition 3.10. Let $(P \in X, D)$ be a germ of a stable pair. Let N be the index of X and $Z \to X$ the canonical covering, a μ_N quotient. Let D_Z denote the inverse image of D. We say a deformation $(\mathcal{X}, \mathcal{D})/(0 \in S)$ of (X, D) is $\mathbb{Q}$-*Gorenstein* if there is a μ_N equivariant deformation $(\mathcal{Z}, \mathcal{D}_\mathcal{Z})/S$ of (Z, D_Z) whose quotient is $(\mathcal{X}, \mathcal{D})/S$.

If $(\mathcal{X}, \mathcal{D})/S$ is a $\mathbb{Q}$-Gorenstein family of stable pairs and $\pi \colon \mathcal{Z} \to \mathcal{X}/S$ is a local canonical covering of $\mathcal{X}/S$, then the closed subscheme $\mathcal{D}_\mathcal{Z} \hookrightarrow \mathcal{Z}$ is uniquely determined by $\mathcal{D} \hookrightarrow \mathcal{X}$. For, the ideal sheaf of $\mathcal{D}_\mathcal{Z}$ in $\mathcal{Z}$ is S_2 over S and agrees with the pullback of the ideal sheaf of $\mathcal{D}$ in $\mathcal{X}$ over the locus where π is étale. Thus $\mathcal{D} \hookrightarrow \mathcal{X}$ defines a closed substack $\mathfrak{D} \hookrightarrow \mathfrak{X}$, where $\mathfrak{X}$ is the canonical covering stack of $\mathcal{X}/S$.

We first show that the families constructed in Theorem 2.12 satisfy the $\mathbb{Q}$-Gorenstein condition. This is needed to prove that the moduli space of stable pairs is proper.

Lemma 3.11. *Let $(\mathcal{X}, \mathcal{D})/(0 \in T)$ be a flat family of stable pairs over the germ of a curve. Suppose that the general fibre of $\mathcal{X}/T$ is smooth and that $K_\mathcal{X}$ and $\mathcal{D}$ are $\mathbb{Q}$-Cartier. Then $(\mathcal{X}, \mathcal{D})/T$ is $\mathbb{Q}$-Gorenstein.*

Proof. The family $\mathcal{X}/T$ is $\mathbb{Q}$-Gorenstein by Lemma 3.4. Working locally at $P \in X \subset \mathcal{X}$, write

$$\begin{array}{ccc} (Z, D_Z) & \subset & (\mathcal{Z}, \mathcal{D}_\mathcal{Z}) \\ \downarrow & & \downarrow \\ (X, D) & \subset & (\mathcal{X}, \mathcal{D}) \\ \downarrow & & \downarrow \\ 0 & \in & T \end{array}$$

for the canonical coverings together with the inverse images of the divisors D and $\mathcal{D}$. We need to show that $\mathcal{D}_{\mathcal{Z}}$ is a deformation of D_Z. We know that $\mathcal{D}_{\mathcal{Z}}$ is $\mathbb{Q}$-Cartier and D_Z is Cartier by Lemma 3.13; it follows that $\mathcal{D}_{\mathcal{Z}}$ is Cartier (cf. [KSB], Lemma 3.16, p. 316) and thus $\mathcal{D}_{\mathcal{Z}} \otimes k(0) = D_Z$ as required. $\square$

Theorem 3.12. *Let $(\mathcal{X}, \mathcal{D})/A$ be a $\mathbb{Q}$-Gorenstein family of stable pairs. Let $A' \to A$ be an infinitesimal extension and $\mathcal{X}'/A'$ a $\mathbb{Q}$-Gorenstein deformation of $\mathcal{X}/A$. Then there exists a $\mathbb{Q}$-Gorenstein deformation $(\mathcal{X}', \mathcal{D}')/A'$ of $(\mathcal{X}, \mathcal{D})/A$.*

Proof. Let $\mathfrak{X}/A$ and $\mathfrak{X}'/A'$ denote the canonical covering stacks of $\mathcal{X}/A$ and $\mathcal{X}'/A'$, and let $\mathfrak{D} \hookrightarrow \mathfrak{X}$ be the closed substack determined by $\mathcal{D} \hookrightarrow \mathcal{X}$. We show that $\mathfrak{D} \hookrightarrow \mathfrak{X}$ deforms to a closed substack $\mathfrak{D}' \hookrightarrow \mathfrak{X}'$; we then obtain the desired deformation $\mathcal{D}' \hookrightarrow \mathcal{X}'$ of $\mathcal{D} \hookrightarrow \mathcal{X}$ by forming the coarse moduli space. By induction, we may assume that the kernel M of $A' \to A$ satisfies $M^2 = 0$. Then the obstruction to deforming $\mathfrak{D} \hookrightarrow \mathfrak{X}$ to a closed substack $\mathfrak{D}' \hookrightarrow \mathfrak{X}'$ lies in $\operatorname{Ext}^2(L_{\mathfrak{D}/\mathfrak{X}}, \mathcal{O}_{\mathfrak{D}} \otimes_A M)$ by [I2], Theorem 1.7. To complete the proof, we compute that this obstruction group is trivial. The ideal sheaf $\mathcal{I}$ of $\mathfrak{D}$ in $\mathfrak{X}$ is locally trivial, i.e., $\mathfrak{D}$ is a Cartier divisor on $\mathfrak{X}$. For, let $\mathcal{Z} \to \mathcal{X}$ be a local canonical covering of $\mathcal{X}/A$ and let $\mathcal{D}_{\mathcal{Z}} \hookrightarrow \mathcal{Z}$ be the closed subscheme corresponding to $\mathfrak{D} \hookrightarrow \mathfrak{X}$. Then $\mathcal{D}_{\mathcal{Z}}$ is flat over A and has Cartier fibres by Lemma 3.13, hence $\mathcal{D}_{\mathcal{Z}}$ is Cartier. In particular, the embedding $\mathfrak{D} \hookrightarrow \mathfrak{X}$ is a local complete intersection, thus $L_{\mathfrak{D}/\mathfrak{X}}$ is isomorphic to $\mathcal{I}/\mathcal{I}^2[-1]$ in the derived category of $\mathfrak{D}$, by [I2], p.160. Thus

$$\operatorname{Ext}^2(L_{\mathfrak{D}/\mathfrak{X}}, \mathcal{O}_{\mathfrak{D}} \otimes_A M) \cong \operatorname{Ext}^1(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_{\mathfrak{D}} \otimes_A M).$$

Now $\mathcal{I}/\mathcal{I}^2$ is locally trivial, hence $\mathcal{E}xt^1(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_{\mathfrak{D}} \otimes_A M) = 0$ and

$$\operatorname{Ext}^1(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_{\mathfrak{D}} \otimes_A M) = H^1(\mathfrak{D}, \mathcal{H}om(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_{\mathfrak{D}} \otimes_A M)).$$

Next, we have

$$\begin{aligned} H^1(\mathfrak{D}, \mathcal{H}om(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_{\mathfrak{D}} \otimes_A M)) &= H^1(\mathfrak{X}, \mathcal{H}om(\mathcal{I}, \mathcal{O}_{\mathfrak{D}} \otimes_A M)) \\ &= H^1(\mathcal{X}, p_\star \mathcal{H}om(\mathcal{I}, \mathcal{O}_{\mathfrak{D}} \otimes_A M)) \end{aligned}$$

where p is the induced map $\mathfrak{X} \to \mathcal{X}$. By cohomology and base change for $\mathcal{X}/A$, we may reduce to the case $A = M = \mathbb{C}$; write $(X, D) = (\mathcal{X}, \mathcal{D})$. Applying $p_\star \mathcal{H}om(\mathcal{I}, -)$ to the exact sequence

$$0 \to \mathcal{I} \to \mathcal{O}_{\mathfrak{X}} \to \mathcal{O}_{\mathfrak{D}} \to 0$$

of sheaves on $\mathfrak{X}$, we obtain the exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(D) \to p_\star \mathcal{H}om(\mathcal{I}, \mathcal{O}_{\mathfrak{D}}) \to 0$$

of sheaves on X. Note that $\mathcal{H}om(\mathcal{I}, -)$ is exact since $\mathcal{I}$ locally free, and $p_\star$ is also exact. Consider the associated long exact sequence of cohomology

$$\cdots \to H^1(\mathcal{O}_X(D)) \to H^1(p_\star \mathcal{H}om(\mathcal{I}, \mathcal{O}_{\mathfrak{D}})) \to H^2(\mathcal{O}_X) \to \cdots .$$

We have $H^1(\mathcal{O}_X(D)) = 0$ by Lemma 3.14 and $H^2(\mathcal{O}_X) = H^0(K_X)^\vee = 0$ by Serre duality and ampleness of $-K_X$. So $H^1(p_\star \mathcal{H}om(\mathcal{I}, \mathcal{O}_{\mathfrak{D}})) = 0$ as required. $\square$

Lemma 3.13. *Let (X, D) be a stable pair and (Z, D_Z) a local canonical covering together with the inverse image of D. Then the divisor D_Z is Cartier.*

Proof. If d is divisible by 3 then $\frac{d}{3}K_X + D \sim 0$, so $D_Z \sim -\frac{d}{3}K_Z \sim 0$. Otherwise, by Theorem 7.1 and Propositions 6.1 and 6.2, the only possible singularities of X are of the forms:

(1) $\frac{1}{n^2}(1, na-1)$, where $3 \not| n$ and $(a, n) = 1$.

(2) $(xy = 0) \subset \frac{1}{r}(1, -1, a)$, where $(a, r) = 1$.

In case (1), the local class group of X is $\mathbb{Z}/n^2\mathbb{Z}$. So, since $dK_X + 3D \sim 0$ and $3 \not| n$, the divisor D is locally a multiple of K_X, hence D_Z is Cartier. In case (2), the local class group of $\mathbb{Q}$-Cartier divisors is $\mathbb{Z}/r\mathbb{Z}$, generated by K_X, so D_Z is Cartier $\square$

Lemma 3.14. *Let (X, D) be a stable pair. Then $H^1(\mathcal{O}_X(D)) = 0$.*

Proof. We have $H^1(\mathcal{O}_X(D)) = H^1(\mathcal{O}_X(K_X - D))^\vee$ by Serre duality and $-(K_X - D)$ is ample. So if X is log terminal our result follows by Kodaira vanishing.

Otherwise, let $\nu\colon X^\nu \to X$ be the normalisation of X and $\tilde{\Delta} \to \Delta$ the normalisation of the double curve Δ. Then for any $\mathbb{Q}$-Cartier divisor E on X there is an exact sequence

$$0 \to \mathcal{O}_X(E) \to \mathcal{O}_{X^\nu}(\nu^\star E) \to \mathcal{O}_{\tilde{\Delta}}(\lfloor E|_{\tilde{\Delta}} \rfloor)$$

and hence a short exact sequence

$$0 \to \mathcal{O}_X(E) \to \mathcal{O}_{X^\nu}(\nu^\star E) \to \mathcal{F} \to 0$$

where $\mathcal{F} \hookrightarrow \mathcal{O}_{\tilde{\Delta}}(\lfloor E|_{\tilde{\Delta}} \rfloor)$. Putting $E = K_X - D$, we have $H^0(\mathcal{F}) = 0$ since $-E$ is ample and $H^1(\mathcal{O}_{X^\nu}(\nu^\star E)) = 0$ if X^ν is log terminal by Kodaira vanishing. So, in this case, the long exact sequence of cohomology gives $H^1(\mathcal{O}_X(K_X - D)) = 0$ as required.

If X^ν is not log terminal, then X is an elliptic cone by Theorems 5.5 and 8.5, the degree d is divisible by 3 and $D \sim -\frac{d}{3}K_X$. An easy calculation shows that $H^1(\mathcal{O}_X(D)) = 0$ in this case. □

4 The moduli space of stable pairs

We construct the moduli space $\mathcal{M}_d$ of stable pairs of degree d using the deformation theory of Section 3.

Definition 4.1. Let $(X, D)/\mathbb{C}$ be a stable pair of degree d. Let $(\mathcal{X}^u, \mathcal{D}^u) \to (0 \in S_0)$ be a versal $\mathbb{Q}$-Gorenstein deformation of the pair $(X, D)/\mathbb{C}$, where S_0 is of finite type over $\mathbb{C}$. Let $S_1 \subset S_0$ be the open subscheme where the fibres of $\mathcal{X}^u/S_0$ are isomorphic to $\mathbb{P}^2$ and let S_2 be the scheme theoretic closure of S_1 in S_0. A $\mathbb{Q}$-Gorenstein deformation of (X, D) is *smoothable* if it is obtained by pullback from the deformation $(\mathcal{X}^u, \mathcal{D}^u) \times_{S_0} S_2 \to (0 \in S_2)$.

Remark 4.2. This definition is vacuous if the degree is not a multiple of 3, i.e., any $\mathbb{Q}$-Gorenstein deformation of (X, D) is automatically smoothable. For (X, D) has unobstructed $\mathbb{Q}$-Gorenstein deformations if $3 \not| d$ by Theorem 7.2, so that, in the notation above, the germ $0 \in S_0$ is smooth and thus the open subscheme $S_1 \subset S_0$ is dense and $S_2 = S_0$. However, if d is a multiple of 3, there are examples where S_0 is reducible and S_2 is an irreducible component of S_0.

Definition 4.3. Let $\underline{Sch}$ be the category of noetherian schemes over $\mathbb{C}$. Let $d \in \mathbb{N}$, $d \geq 4$. We define a stack $\mathcal{M}_d \to \underline{Sch}$ as follows:

$$\mathcal{M}_d(S) = \left\{ (\mathcal{X}, \mathcal{D})/S \;\middle|\; \begin{array}{c} (\mathcal{X}, \mathcal{D})/S \text{ is a } \mathbb{Q}\text{-Gorenstein smoothable} \\ \text{family of stable pairs of degree } d \end{array} \right\}$$

Theorem 4.4. *The stack $\mathcal{M}_d$ is a separated and proper Deligne–Mumford stack. The underlying coarse moduli space is a compactification of the moduli space of smooth plane curves of degree d.*

We give the salient points in the proof of the theorem. Using the obstruction theory for $\mathbb{Q}$-Gorenstein deformations obtained in Section 3, we deduce the existence of versal $\mathbb{Q}$-Gorenstein deformations for stable pairs, corresponding to local patches of the stack $\mathcal{M}_d$ [Ar]. To prove boundedness, i.e., that only finitely many patches are required, we first bound the index of a surface X occurring in a stable pair of degree d (Theorem 4.5). Then, letting $N(d) \in \mathbb{N}$ be such that $N(d)K_X$ is Cartier for each such pair (X, D), we have a polarisation on each (X, D) given by $-N(d)K_X$. The Hilbert polynomial is fixed by our smoothability assumption; hence boundedness follows by [Ko], Theorem 2.1.2. Finally, the stack $\mathcal{M}_d$ is separated and proper by Theorem 2.12.

Theorem 4.5. *Let (X, D) be a stable pair of degree d. Then the index of each point $P \in X$ is at most d. Moreover, the same result holds if (X, D) is a semistable pair of degree d and d is not a multiple of 3.*

Proof. The pair $(X, (\frac{3}{d} + \epsilon)D)$ is slc, hence D misses the strictly slc points of X. Then the condition $dK_X + 3D \sim 0$ shows that the index of X is at most d at such points.

The slt singularities of X are of the following types (by Propositions 6.1 and 6.2):

(1) $\frac{1}{n^2}(1, na - 1)$, where $(a, n) = 1$ and $3 \not| n$.

(2) $(xy = 0) \subset \frac{1}{r}(1, -1, a)$ where $(a, r) = 1$.

(3) $(x^2 = zy^2) \subset \mathbb{A}^3$.

The index of X equals n, r and 1 in cases (1), (2) and (3) respectively.

In case (1) let $\tilde{X} \to X$ be the local smooth covering of X and $\tilde{D}$ the inverse image of D. Write $\tilde{X} = \mathbb{A}^2_{x,y}$ and $\tilde{D} = (f(x, y) = 0)$. The multiplicity of the divisor $\tilde{D}$ at $0 \in \tilde{X}$ is strictly less than $\frac{2d}{3}$ since $(\tilde{X}, (\frac{3}{d} + \epsilon)\tilde{D})$ is log canonical. Let $x^i y^j$ be a monomial appearing in the polynomial $f(x, y)$ such that $i+j$ is minimal, thus $i+j < \frac{2d}{3}$. Then $3(i+(na-1)j) = dna \mod n^2$, using $dK_X + 3D \sim 0$. In particular, $i = j \mod n$. Thus if $n > d$ then $i = j < \frac{d}{3}$ and $3i = d \mod n$, a contradiction.

In case (2) let $\tilde{X} \to X$ be the canonical covering of X, let $\tilde{D}$ denote the inverse image of D and $\tilde{\Delta}$ the inverse image of the double curve of X. Write $\tilde{X} = (xy = 0) \subset \mathbb{A}^3_{x,y,z}$ and $\tilde{D} = (f(x, y, z) = 0)$. Then

$$\tilde{D}|_{\tilde{\Delta}} = (f(0, 0, z) = 0) = (z^k + \cdots = 0) \subset \mathbb{A}^1_z,$$

where k is the multiplicity of $\tilde{D}|_{\Delta}$ at $0 \in \tilde{\Delta}$. Then $k < \frac{d}{3}$ since $(X, (\frac{3}{d}+\epsilon)D)$ is slc and $3k = d \mod r$ since $dK_X + 3D \sim 0$. Hence $r \leq d$ as required.

If (X, D) is a semistable pair of degree d, then X has only singularities of type $\frac{1}{n^2}(1, na - 1)$, the pair $(X, \frac{3}{d}D)$ is log canonical and $dK_X + 3D \sim 0$. Then, assuming $3 \not| d$, proceeding as in case (1) above we deduce the same result. $\square$

5 A coarse classification of the degenerate surfaces

If (X, D) is a stable pair then the surface X is slc and the divisor $-K_X$ is ample. We use these two properties to obtain a coarse classification of the possible surfaces X. We first describe the pairs (Y, C) where Y is an irreducible component of the normalisation of X and C is the inverse image of the double curve of X. We then glue such pairs together to obtain the classification of the surfaces X.

Theorem 5.1. *([KM], p. 119, Theorem 4.15) Let $P \in Y$ be the germ of a surface and C an effective divisor on Y such that the pair (Y, C) is log canonical. Then, assuming $C \neq 0$, the germ $(P \in Y, C)$ is of one of the following types:*

(1) $(\frac{1}{r}(1,a), (x=0))$*, where* $(a,r)=1$.

(2) $(\frac{1}{r}(1,a), (xy=0))$*, where* $(a,r)=1$.

(3) $(\frac{1}{r}(1,a), (xy=0))/\mu_2$*, where the* μ_2 *action is etale in codimension* 1 *and interchanges* $(x=0)$ *and* $(y=0)$.

Moreover (1) is log terminal, whereas (2) and (3) are strictly log canonical.

Notation 5.2. We denote singularities of types (1), (2) and (3) by $(\frac{1}{r}(1,a), \Delta)$, $(\frac{1}{r}(1,a), 2\Delta)$ and (D, Δ) respectively. The D stands for dihedral — the surface singularities $P \in Y$ here include the dihedral Du Val singularities.

Theorem 5.3. *Let Y be a surface and C an effective divisor on Y such that the pair (Y, C) is log canonical and $-(K_Y + C)$ is ample. Then (Y, C) is of one of the following types:*

(I) $C = 0$.

(II) $C \cong \mathbb{P}^1$ *and* (Y, C) *is log terminal.*

(III) $C \cong \mathbb{P}^1 \cup \mathbb{P}^1$*, where the components meet in a single node.*

(IV) $C \cong \mathbb{P}^1$ *and* (Y, C) *has a singularity of type* (D, Δ).

Moreover, in case (I) the surface Y has at most one strictly log canonical singularity, in case (III) the pair (Y, C) is log terminal away from the node of C and in case (IV) the pair (Y, C) is log terminal away from the singularity of type (D, Δ).

Proof. The pair (Y, C) is log canonical and $-(K_Y + C)$ is ample by assumption, hence the locus where (Y, C) is not klt is connected by the connectedness theorem of Kollár and Shokurov (cf. [KM], p. 173, Theorem 5.48 and Corollary 5.49). In other words, either $C = 0$ and Y has at most one strictly log canonical singularity, or C is connected and Y is log terminal away from C.

If $C \neq 0$, let Γ be a component of C. Then

$$(K_Y + C)\Gamma = (K_Y + \Gamma)\Gamma + (C - \Gamma)\Gamma = 2p_a(\Gamma) - 2 + \mathrm{Diff}(Y, \Gamma) + (C - \Gamma)\Gamma$$

where $\mathrm{Diff}(Y, \Gamma)$ is the different of the pair (Y, Γ), i.e., the correction to the adjunction formula for $\Gamma \subset Y$ required due to the singularities of Y at Γ ([FA], Chapter 16). Now $(K_Y + C)\Gamma < 0$ since $-(K_Y + C)$ is ample, the different $\mathrm{Diff}(Y, \Gamma) \geq 0$ and $(C - \Gamma)\Gamma \geq 0$. So $p_a(\Gamma) = 0$, i.e., the curve Γ is smooth and rational, and

$$\mathrm{Diff}(Y, \Gamma) + (C - \Gamma)\Gamma < 2.$$

The singularities of (Y, C) at Γ are of the forms $(\frac{1}{r}(1, a), \Delta)$, $(\frac{1}{r}(1, a), 2\Delta)$ and (D, Δ) as described in Theorem 5.1. We calculate that these singularities contribute $1 - \frac{1}{r}$, 1 and 1 to the value of $\mathrm{Diff}(Y, \Gamma) + (C - \Gamma)\Gamma$ respectively. The theorem now follows easily. $\square$

Notation 5.4. Let X be an slc surface. Let Δ denote the double curve of X. Let $X_1, \ldots, X_n$ be the irreducible components of X and write Δ_i for the restriction of Δ to X_i. Let $\nu \colon X^\nu \to X$ be the normalisation of X and Δ^ν the inverse image of Δ; also write X_i^ν for the normalisation of X_i and Δ_i^ν for the inverse image of Δ_i.

The map $\Delta^\nu \to \Delta$ is 2-to-1. Let $\Gamma \subset \Delta$ be a component and write Γ^ν for its inverse image on X^ν. Then either Γ^ν has two components mapping birationally to Γ or Γ^ν is irreducible and is a double cover of Γ. In the latter case we say that the curve $\Gamma^\nu \subset X^\nu$ is *folded* to obtain $\Gamma \subset X$.

Theorem 5.5. *Let X be a slc surface such that $-K_X$ is ample. Then X is of one of the following types:*

(A) X is normal.

(B) X has two normal components meeting in a smooth rational curve and is slt.

(B) X is irreducible, non-normal and slt. The pair (X^ν, Δ^ν) is of type II and X is obtained by folding the curve Δ^ν.*

(C) *X has n components $X_1, \cdots, X_n$ such that (X_i^ν, Δ_i^ν) is of type III for each i. One component of Δ_i^ν is glued to a component of $\Delta_{i+1 \mod n}^\nu$ for each i so that the nodes of the curves Δ_i^ν coincide and the components X_i of X form an 'umbrella'.*

(D) *X has n components $X_1, \ldots, X_n$ such that (X_i, Δ_i) is of type III for $2 \leq i \leq n-1$. Either (X_1, Δ_1) is of type IV or (X_1^ν, Δ_1^ν) is of type III and (X_1, Δ_1) is obtained by folding one component of Δ_1^ν; similiarly for (X_n, Δ_n). The components $(X_1, \Delta_1), \ldots, (X_n, \Delta_n)$ are glued sequentially so that the nodes of the curves Δ_i and any (D, Δ) singularities on (X_1, Δ_1) and (X_n, Δ_n) coincide and the components X_i of X form a 'fan'.*

Proof. Let (Y, C) be a component of the pair (X^ν, Δ^ν), then (Y, C) is log canonical and $-(K_Y + C)$ is ample since X is slc and $-K_X$ is ample. Hence (Y, C) is of one of the types I–IV described in Theorem 5.3. Glueing the components back together (using the classification of slc singularities [KSB]) we obtain the classification of the surfaces X given above. □

Theorem 5.6. *Let (Y, C) be as in Theorem 5.3. Then either Y is rational or $C = 0$ and Y is an elliptic cone.*

Proof. Let $\pi \colon \tilde{Y} \to Y$ be the minimal resolution of Y. Let $\tilde{C}$ be the $\mathbb{Q}$-divisor defined by the equation

$$K_{\tilde{Y}} + \tilde{C} = \pi^\star(K_Y + C).$$

Note that $\tilde{C}$ is effective since π is minimal and $-(K_{\tilde{Y}} + \tilde{C})$ is nef and big since $-(K_Y + C)$ is ample. So, running a MMP, we obtain a birational morphism $\phi \colon \tilde{Y} \to Y_1$ where either $Y_1 \cong \mathbb{P}^2$ or Y_1 has the structure of a $\mathbb{P}^1$-bundle $q \colon Y_1 \to B$ over a smooth curve B. We may assume that Y is not rational, so that we are in the second case and the curve B has positive genus.

We claim that there is an irrational component of the divisor $\tilde{C}$. For, otherwise, the image $C_1 = \phi_\star \tilde{C}$ is a sum of fibres of the ruling. Then $-(K_{Y_1} + C_1)$ nef and big implies that $-K_{Y_1}$ is nef and big, hence $h^1(\mathcal{O}_{Y_1}) = 0$ by Kodaira vanishing. So B has genus zero, a contradiction.

We have $\operatorname{Supp} \tilde{C} \subset C' \cup \operatorname{Ex}(\pi)$ where C' denotes the strict transform of C on $\tilde{Y}$ and $\operatorname{Ex}(\pi)$ is the exceptional locus of π. By Theorem 5.3 the curve C has only rational components, so $\operatorname{Ex}(\pi)$ contains an irrational curve and Y has a simple elliptic singularity by the classification of log canonical singularities. Let E denote the corresponding π-exceptional elliptic curve on $\tilde{Y}$. Then E has multiplicity 1 in $\tilde{C}$ and is horizontal with respect to the birational ruling $\tilde{Y} \to B$. The divisor $-(K_{\tilde{Y}} + \tilde{C})$ is big, so $-(K_{\tilde{Y}} + \tilde{C})f > 0$ where f is a fibre of the ruling. Hence $E \cdot f = 1$, i.e., the curve E is a section of the ruling.

We show that $\tilde{Y}$ is actually biregularly ruled over the elliptic curve E. Suppose not, then there is a degenerate fibre; write A for a component meeting E. Then A is not contained in $\operatorname{Supp} \tilde{C}$ and $(K_{\tilde{Y}} + \tilde{C})A \leq 0$, with equality if and only if A is contracted by π. But also

$$(K_{\tilde{Y}} + \tilde{C})A \geq K_{\tilde{Y}}A + E \cdot A \geq -1 + 1 = 0$$

with equality only if A is a (-1)-curve. So A is a (-1)-curve which is contracted by π, a contradiction since π is minimal.

Thus $\tilde{Y}$ is a $\mathbb{P}^1$-bundle over an elliptic curve and the surface Y is obtained by contracting the negative section; so Y is an elliptic cone. Finally $C = 0$ by Theorem 5.3. □

6 Preliminary smoothability results

We collect some further restrictions on the degenerate surfaces X implied by the existence of a smoothing to the plane. We give a more detailed analysis of the possible singularities and some restrictions on the Picard numbers of the components of X. We deduce that a surface of type B* cannot smooth to the plane.

Proposition 6.1. *([KSB], Theorem 4.23 and 5.2) Let $P \in X$ be an slt surface singularity which admits a $\mathbb{Q}$-Gorenstein smoothing. Then $P \in X$ is of one of the following types:*

(1) A Du Val singularity.

(2) $\frac{1}{dn^2}(1, dna - 1)$, where $(a, n) = 1$.

(3) $(xy = 0) \subset \frac{1}{r}(1, -1, a)$, where $(a, r) = 1$.

(4) $(x^2 = y^2 z) \subset \mathbb{A}^3$.

Proposition 6.2. *Let X be an slc surface which admits a $\mathbb{Q}$-Gorenstein smoothing to $\mathbb{P}^2$. Then the log terminal singularities of X are of the form $\frac{1}{n^2}(1, na - 1)$, where $3 \not| n$.*

Proof. In the case that X is globally log terminal this was proved in [Ma], Section 3. The same argument proves our result. □

Proposition 6.3. *Let X be an slc surface such that $-K_X$ is ample. Suppose that X admits a smoothing to $\mathbb{P}^2$, i.e., there exists a flat family $\mathcal{X}/(0 \in T)$ over the germ of a curve with special fibre X and general fibre $\mathbb{P}^2$. Then, in the cases A, B and B* of Theorem 5.5 the Picard numbers of the components of X are as follows:*

(A) $\rho(X) = 1$.

(B) Either (i) $\rho(X_1) = \rho(X_2) = 1$ *or (ii)* $\{\rho(X_1), \rho(X_2)\} = \{1, 2\}$.

(B)* $\rho(X^\nu) = 1$.

Moreover, given a smoothing $\mathcal{X}/T$ of X as above, the total space $\mathcal{X}$ is $\mathbb{Q}$-factorial unless X is of type B, case (i).

Remark 6.4. In fact a surface of type B* never admits a smoothing to $\mathbb{P}^2$ by Theorem 6.5 below — the above result is required in the proof.

Proof. Consider the commutative diagram

$$\begin{array}{ccc} \operatorname{Pic}\mathcal{X} & \to & \operatorname{Pic} X \\ \downarrow & & \downarrow \\ H^2(\mathcal{X}, \mathbb{Z}) & \to & H^2(X, \mathbb{Z}) \end{array}$$

We claim that all these maps are isomorphisms. First, the restriction map $H^2(\mathcal{X}, \mathbb{Z}) \to H^2(X, \mathbb{Z})$ is an isomorphism because X is a homotopy retract of $\mathcal{X}$. Second, the map $c_1\colon \operatorname{Pic} X \to H^2(X, \mathbb{Z})$ fits into the long exact sequence of cohomology

$$\cdots \to H^1(\mathcal{O}_X) \to \operatorname{Pic} X \to H^2(X, \mathbb{Z}) \to H^2(\mathcal{O}_X) \to \cdots$$

associated to the exponential sequence on X. Now $H^2(\mathcal{O}_X) = 0$ since $-K_X$ is ample, so also $H^1(\mathcal{O}_X) = 0$ since $\chi(\mathcal{O}_X) = \chi(\mathcal{O}_{\mathbb{P}^2})$. Thus c_1 is an isomorphism as claimed. Similiarly $c_1\colon \operatorname{Pic}\mathcal{X} \to H^2(\mathcal{X}, \mathbb{Z})$ is also an isomorphism. Hence the restriction map $\operatorname{Pic}(\mathcal{X}) \to \operatorname{Pic}(X)$ is an isomorphism. Now $\operatorname{Cl}(\mathcal{X}) \cong \mathbb{Z}^n$ by Lemma 2.11, so we have the inequality

$$\rho(X) = \dim \operatorname{Pic}(X) \otimes \mathbb{Q} = \dim \operatorname{Pic}(\mathcal{X}) \otimes \mathbb{Q} \leq \dim \operatorname{Cl}(\mathcal{X}) \otimes \mathbb{Q} = n,$$

with equality if and only if $\mathcal{X}$ is $\mathbb{Q}$-factorial.

We next relate the Picard numbers of X and its irreducible components. Note that $\rho(X) = \dim H_2(X, \mathbb{Q})$ by the above and similiarly for the components of X, since they are rational by Theorem 5.6. If X is a surface of type B then the Mayer-Vietoris sequence for $X = X_1 \cup X_2$ yields the following exact sequence:

$$\begin{array}{ccccccccc} 0 & \to & \mathbb{Q} & \to & H_2(X_1,\mathbb{Q}) \oplus H_2(X_2,\mathbb{Q}) & \to & H_2(X) & \to & 0 \\ & & 1 & \mapsto & \Delta \oplus \Delta & & & & \end{array}$$

So $\rho(X) = \rho(X_1) + \rho(X_2) - 1$. If X is a surface of type B* then an easy Mayer-Vietoris argument shows that $H_2(X^\nu) \cong H_2(X)$, so $\rho(X^\nu) = \rho(X)$. Our result now follows easily using the inequality $\rho(X) \leq n$ derived above. $\square$

Theorem 6.5. *Let X be a surface of type B*. Then X does not admit a smoothing to $\mathbb{P}^2$.*

Proof. Suppose X is a counter-example and let $\mathcal{X}/T$ be a smoothing of X to $\mathbb{P}^2$. Then $\rho(X^\nu) = 1$ and $\mathcal{X}$ is $\mathbb{Q}$-factorial by Proposition 6.3; in particular $K_{\mathcal{X}}$ is $\mathbb{Q}$-Cartier. Thus

$$(K_{X^\nu} + \Delta^\nu)^2 = (\nu^\star K_X)^2 = K_X^2 = K_{\mathbb{P}^2}^2 = 9$$

and so $K_{X^\nu}^2 > 9$, since $-K_X$ is ample and $\rho(X^\nu) = 1$. Hence

$$K_{X^\nu}^2 + \rho(X^\nu) > 10.$$

On the other hand, let $\tilde{X}$ be the minimal resolution of X^ν, then $K_{\tilde{X}}^2 + \rho(\tilde{X}) = 10$ by Noether's formula, since $\tilde{X}$ is rational by Theorem 5.6. We calculate below that the only possible singularities on X^ν cause an increase in $K^2 + \rho$ on passing to the minimal resolution, so we have a contradiction. In fact, if a *normal* rational surface singularity $P \in S$ admits a $\mathbb{Q}$-Gorenstein smoothing then, writing $\tilde{S} \to S$ for minimal resolution, the Milnor number μ of the smoothing equals $K_{\tilde{S}}^2 + \rho(\tilde{S})$ (cf. [Lo]). In particular, $K_{\tilde{S}}^2 + \rho(\tilde{S})$ is a non-negative integer.

The pair (X^ν, Δ^ν) has singularities of types $(\frac{1}{n^2}(1, na-1), 0)$ and $(\frac{1}{r}(1,a), \Delta)$, with the latter cases occurring in pairs $\frac{1}{r}(1,a)$ and $\frac{1}{r}(-1,a)$, by Proposition 6.1 and Proposition 6.2. Given a cyclic quotient singularity $\frac{1}{r}(1,a)$, let $\frac{r}{a} = [b_1, \ldots, b_k]$ be the expansion of $\frac{r}{a}$ as a Hirzebruch–Jung continued fraction ([Fu], pp. 45-7). Then the minimal resolution of the singularity has exceptional locus a chain of smooth rational curves $E_1, \ldots, E_n$ with self intersections $-b_1, \ldots, -b_k$. On passing to the minimal resolution, the change in $K^2 + \rho$ is given by

$$\delta = E^2 + 4 - \frac{1}{r}(a + a' + 2)$$

where $E = E_1 + \cdots + E_n$ and a' is the inverse of a modulo r. For the singularity $\frac{1}{n^2}(1, na-1)$ we calculate $\delta = 0$ using the inductive description of the minimal resolutions of these singularities (see [KSB], p. 314, Proposition 3.11). For a pair of singularities $\frac{1}{r}(1,a)$, $\frac{1}{r}(1,-a)$ we calculate $\delta_1 + \delta_2 = 4(1-\frac{1}{r})$. Here we use the following elementary fact: if $\frac{r}{a} = [b_1, \ldots, b_k]$ and $\frac{r}{r-a} = [c_1, \ldots, c_l]$, then

$$\sum (b_i - 1) = \sum (c_j - 1) = k + l - 1.$$

□

7 Simplifications in the case $3 \not| d$

We show how the classification of stable pairs of degree d simplifies considerably if d is not a multiple of 3. We deduce that the stack $\mathcal{M}_d$ is smooth in this case.

Theorem 7.1. *Let (X, D) be a stable pair of degree d, where d is not a multiple of* 3. *Then X is slt. So X is either a normal log terminal surface or a surface of type B. In particular, the surface X has either* 1 *or* 2 *components.*

Proof. Since $3 \not| d$, the condition $dK_X + 3D \sim 0$ gives an arithmetic condition on X, namely that K_X is 3-divisible as a $\mathbb{Q}$-Cartier divisor class on X. Roughly, given a curve Γ on X we have $dK_X \cdot \Gamma = 3D \cdot \Gamma$, so $K_X \cdot \Gamma$ should be divisible by 3. Of course the intersection number $D \cdot \Gamma$ is not an integer in general since X is singular, but this is the idea of the proof.

First suppose that X is normal. Then either X is log terminal or X is an elliptic cone by Theorem 8.5. In the second case, let Γ be a ruling of the cone, then $3D \cdot \Gamma = -dK_X \cdot \Gamma = d$. But D misses the singularity of X since the pair $(X, (\frac{3}{d} + \epsilon)D)$ is log canonical, hence D is Cartier and $D \cdot \Gamma$ is an integer, a contradiction.

Now suppose X is not normal. Let (Y, C) be a component of the pair (X^ν, Δ^ν), where X^ν is the normalisation of X and Δ^ν is the inverse image of the double curve of X. We need to show that (Y, C) is log terminal. Suppose this is not the case, then the pair (Y, C) has a singularity of type $(\frac{1}{r}(1,a), 2\Delta)$ or (D, Δ). Let Γ be a component of C passing through such a point. Then there is at most one other singularity of (Y, C) on Γ, of type $(\frac{1}{s}(1,b), \Delta)$, and

$$(K_Y + C)\Gamma = -2 + 1 + (1 - \frac{1}{s}) = -\frac{1}{s},$$

cf. Theorem 5.3. We allow $s = 1$, corresponding to the case that there are no further singularities at Γ. Then $3D \cdot \Gamma = -d(K_Y + C)\Gamma = \frac{d}{s}$. But D misses the strictly log canonical singularity of X, hence sD is Cartier near Γ and $sD \cdot \Gamma$ is an integer, a contradiction. $\square$

Theorem 7.2. *The stack $\mathcal{M}_d$ is smooth if $3 \not| d$.*

Proof. Assume $3 \not| d$. Given a stable pair (X, D) of degree d, the surface X is slt by Theorem 7.1. So X is either normal and log terminal or a surface of type B, and X has unobstructed $\mathbb{Q}$-Gorenstein deformations by Theorem 8.2 or Theorem 9.1 respectively. The $\mathbb{Q}$-Gorenstein deformations of the pair (X, D) are thus unobstructed by Theorem 3.12. Hence $\mathcal{M}_d$ is smooth as required. $\square$

8 The normal surfaces

8.1 Log terminal surfaces

Log terminal degenerations of the plane have been classified by Manetti [Ma]. We announce a refinement of his result below (Theorem 8.3), the proof will appear elsewhere.

Definition 8.1. A *Manetti surface* is a normal log terminal surface which smoothes to $\mathbb{P}^2$.

Theorem 8.2. *Let X be a Manetti surface. Then X has unobstructed $\mathbb{Q}$-Gorenstein deformations.*

Proof. The obstructions are contained in $T^2_{QG,X}$ and there is a spectral sequence

$$E_2^{pq} = H^p(\mathcal{T}^q_{QG,X}) \Rightarrow T^{p+q}_{QG,X},$$

hence it is enough to show that $H^p(\mathcal{T}^q_{QG,X}) = 0$ for $p + q = 2$. The sheaf $\mathcal{T}^1_{QG,X}$ is supported on the singular locus, a finite set, so $H^1(\mathcal{T}^1_{QG,X}) = 0$. The singularities of X are of the form $\frac{1}{n^2}(1, na - 1)$ by Proposition 6.2. Let $\pi \colon Z \to X$ be a local canonical covering, with group μ_n. Then Z is a hypersurface, so $\mathcal{T}^2_Z = 0$ and $\mathcal{T}^2_{QG,X} = (\pi_\star \mathcal{T}^2_Z)^{\mu_n} = 0$. Hence $H^0(\mathcal{T}^2_{QG,X}) = 0$. Finally $H^2(\mathcal{T}^0_{QG,X}) = H^2(\mathcal{T}_X) = 0$ by [Ma], p. 113. $\square$

Theorem 8.3. *Let (a, b, c) be a solution of the Markov equation*

$$a^2 + b^2 + c^2 = 3abc.$$

Then the weighted projective space $X = \mathbb{P}(a^2, b^2, c^2)$ smoothes to $\mathbb{P}^2$. Moreover, X has no locally trivial deformations and there is precisely one deformation parameter for each singularity. Conversely, every Manetti surface is obtained as a $\mathbb{Q}$-Gorenstein deformation of such a surface X.

The solutions of the Markov Equation are easily described [Mo]. First, $(1, 1, 1)$ is a solution. Second, given one solution, we obtain another by regarding the equation as a quadratic in one of the variables, c (say), and replacing c by the other root, i.e.,

$$(a, b, c) \mapsto (a, b, 3ab - c).$$

This process is called a *mutation.* All solutions are obtained from $(1, 1, 1)$ by a sequence of mutations. The solutions lie at the vertices of an infinite tree, where two vertices are joined by an edge if they are related by a single mutation. Each vertex has degree 3, and there is a natural action of S_3 on the tree obtained by permuting the variables. The first few solutions are $(1, 1, 1)$, $(1, 1, 2)$, $(1, 2, 5)$, $(1, 5, 13)$, $(2, 5, 29)$. Hence Theorem 8.3 provides a very explicit description of all Manetti surfaces.

8.2 Log canonical surfaces

We show that the log canonical degenerations of the plane are precisely the Manetti surfaces and the elliptic cones of degree 9.

Lemma 8.4. *[Ma] Let X be a normal rational surface which smoothes to $\mathbb{P}^2$. Let $\pi\colon \tilde{X} \to X$ be the minimal resolution of X. Then, assuming X is not isomorphic to $\mathbb{P}^2$, there exists a birational morphism $\tilde{X} \to \mathbb{F}_w$; fix one such morphism μ with w maximal. Let $p\colon \tilde{X} \to \mathbb{P}^1$ denote the birational ruling induced by μ, let B be the negative section of $\mathbb{F}_w$ and B' its strict transform on X. Then the exceptional locus of π consists of the curve B' together with the components of the fibres of p with self intersection at most -2, i.e.,*

$$\mathrm{Ex}(\pi) = B' \cup \{\Gamma \subset \tilde{X} \mid p_\star\Gamma = 0 \text{ and } \Gamma^2 \leq -2\}.$$

Moreover, every degenerate fibre of p contains a unique (-1)-curve.

Theorem 8.5. *Let X be a normal log canonical surface with $-K_X$ ample which admits a smoothing to $\mathbb{P}^2$. Then either X is log terminal or X is an elliptic cone of degree 9.*

Proof. The smoothing of X is necessarily $\mathbb{Q}$-Gorenstein by Propostion 6.3, so $K_X^2 = K_{\mathbb{P}^2}^2 = 9$. If X is not rational, then X is an elliptic cone by Theorem 5.6, and X has degree 9 since $K_X^2 = 9$.

Now suppose that X is rational. We assume that X has a strictly log canonical singularity and obtain a contradiction. We first describe the rational strictly log canonical surface singularities ([KM], p. 112–116). The exceptional locus E of the minimal resolution is a collection of smooth rational curves in one of the following configurations:

(α) $E = G_1 \cup G_2 \cup F_1 \cup \cdots \cup F_k \cup G_3 \cup G_4$, where $F_1 \cup \cdots \cup F_k$ is a chain of smooth rational curves and $G_1, \cdots, G_4$ are (-2)-curves. The curves G_1 and G_2 each intersect F_1 in a single node and similiarly G_3 and G_4 each intersect F_k.

(β) $E = F \cup G^1 \cup G^2 \cup G^3$, where F is a smooth rational curve, each G^i is a chain of smooth rational curves $G^i_1 \cup \cdots \cup G^i_{k(i)}$ and the end component G^i_1 intersects F in a single node.

We use the notation and result of Lemma 8.4. Note first that $\mu\colon \tilde{X} \to \mathbb{F}_w$ is an isomorphism over the negative section B of $\mathbb{F}_w$ by the maximality of w. Consider the MMP yielding $\mu\colon \tilde{X} \to \mathbb{F}_w$ in a neigbourhood of a given degenerate fibre f of the ruling $p\colon \tilde{X} \to \mathbb{P}^1$. At each stage we contract a (-1)-curve which meets at most 2 components of the fibre and is disjoint from B'. By the Lemma, we have

$$\mathrm{Ex}(\pi) = B' \cup \{\Gamma \subset \tilde{X} \mid p_\star \Gamma = 0 \text{ and } \Gamma^2 \leq -2\}.$$

This set decomposes into the exceptional loci of the minimal resolutions of the log canonical singularity and some singularities of type $\frac{1}{n^2}(1, na-1)$. Let E denote the connected component of the exceptional locus contracting to the log canonical singularity. Then E has a component C meeting 3 other components of E — we call such a curve a *fork* of E. We can now describe the form of the degenerate fibre f. Suppose first that f contains a fork of E. Then we have a decompostion $f = P \cup \Gamma \cup Q \cup C \cup R \cup S$, where

(1) The curve Γ is the unique (-1)-curve contained in f and C is a fork of E.

(2) The curve P is either empty or a chain of smooth rational curves, with one end component meeting Γ, which contracts to a singularity of type $\frac{1}{n^2}(1, na-1)$.

(3) The curves Q, R and S are non-empty configurations of smooth rational curves such that Q connects Γ to C and S connects C to B' while R intersects only C.

Let A' denote the component of f meeting B', i.e., the strict transform of the corresponding fibre of $\mathbb{F}_w$. Note that A' cannot be a fork of E since f contains only one (-1)-curve; in particular S is non-empty as claimed. Also, in the MMP $\tilde{X} \to \cdots \to \mathbb{F}_w$, we contract the components of $f - A'$ in the following order: Γ, $P \cup Q$, C, $R \cup S - A'$.

Suppose now that f does not contain a fork of E. Then we have a decompositon $f = P \cup \Gamma \cup Q$, where

(1) The curve Γ is the unique (-1)-curve contained in f.

(2) The curve P is either empty or a chain of smooth rational curves, with one end component meeting Γ, which contracts to a singularity of type $\frac{1}{n^2}(1, na-1)$.

(3) The curve Q is a non-empty chain of curves meeting Γ, with one end component meeting B'.

Suppose that E has the form (α) above. If one of the forks F_1, F_k of E is contained in a degenerate fibre f, then we can write $f = P \cup \Gamma \cup Q \cup C \cup R \cup S$ as above where, without loss of generality, $Q = G_1$, $C = F_1$, $R = G_2$ and $S = F_2 \cup \cdots \cup F_l$ for some $l < k$. Note that E cannot contain the remaining fork F_k of E since then $F_k = A'$, a contradiction. Considering the MMP $\tilde{X} \to \cdots \to \mathbb{F}_w$ again, we deduce that the curves in the chain P have self-intersections $-3, -2, \ldots, -2$. Thus P contracts to an $\frac{1}{2r+1}(1, r)$ singularity, where r is the length of the chain. But this is not of type $\frac{1}{n^2}(1, na-1)$, a contradiction. Hence P is empty. It follows that the curves in the chain $S = F_2 \cup \cdots \cup F_l$ have self-intersections $-3, -2, \ldots, -2, -1$ if $l > 2$. But $F_l^2 \leq -2$, hence $l = 2$ and $F_2^2 = -2$. On the other hand, if the fork F_1 is not contained in a degenerate fibre, then $F_1 = B'$. Then there is a degenerate fibre f of the second form $P \cup \Gamma \cup Q$ with $Q = G_1$, a (-2)-curve. It follows that the chain P is a single (-2)-curve, which contracts to a $\frac{1}{2}(1,1)$ singularity, a contradiction. Combining our results, we deduce that $k = 5$, there are two fibres of the form $\Gamma \cup G_1 \cup G_2 \cup F_1 \cup F_2$ as above and $F_3 = B'$. There are no further degenerate fibres. We compute that $w = 11$ using $K_X^2 = 9$.

We claim that the surface X constructed above does not admit a smoothing to $\mathbb{P}^2$. Let $Z \to X$ be a local canonical covering of the singularity. Then $Z \to X$ is a μ_2 quotient and Z has a cusp singularity. The minimal resolution of Z has exceptional locus a cycle of smooth rational curves with self-intersections $-2, -2, -2, -11, -2, -2, -2, -11$. Suppose there exists a smoothing of X to $\mathbb{P}^2$, then we obtain a smoothing of Z by taking the canonical covering. Let M denote the Milnor fibre of the smoothing of Z and let μ_- denote the number of negative entries in a diagonalisation of the intersection form on $H^2(M,\mathbb{R})$. Then

$$\mu_- = 10h^1(\mathcal{O}_{\tilde{Z}}) + K_{\tilde{Z}}^2 + b_2(\tilde{Z}) - b_1(\tilde{Z})$$

where $\tilde{Z}$ is the minimal resolution of Z [St2]. In our case we calculate $\mu_- = 10 - 18 + 8 - 1 = -1$, a contradiction.

Suppose now that E has the form (β). We first describe E in more detail. The chains G^1, G^2 and G^3 can be contracted to yield a partial resolution $\phi\colon \hat{X} \to X$; write $\hat{F}$ for the image of F on $\hat{X}$. The chains G^i contract to singularities of the pair $(\hat{X},\hat{F})$ of type $(\frac{1}{r}(1,a),\Delta)$. Let r_1, r_2 and r_3 be the indices of these singularities, then $\sum \frac{1}{r_i} = 1$. For X is assumed to be strictly log canonical, hence $K_{\hat{X}} = \phi^\star K_X - \hat{F}$ or, equivalently,

$$0 = (K_{\hat{X}} + \hat{F})\hat{F} = -2 + \sum(1 - \frac{1}{r_i}) = 1 - \sum \frac{1}{r_i}.$$

So $(r_1,r_2,r_3) = (2,3,6)$, $(2,4,4)$ or $(3,3,3)$. In particular, each chain G^i is either a single smooth rational curve of self-intersection $-r_i$ or a chain of (-2)-curves of length $(r_i - 1)$.

We claim that the fork F of E cannot be contained in a fibre f. It is enough to show that w is greater than 2. For ${B'}^2 = B^2 = -w$, so if $w > 2$ then $F = A'$ by the description of E above, a contradiction. Define an effective $\mathbb{Q}$-divisor $\tilde{C}$ on $\tilde{X}$ by $K_{\tilde{X}} + \tilde{C} = \pi^\star K_X$ and let C_1 be the image of $\tilde{C}$ on $\mathbb{F}_w$. Then

$$(K_{\mathbb{F}_w} + C_1)^2 > (K_{\tilde{X}} + \tilde{C})^2 = K_X^2 = 9.$$

Since μ is an isomorphism over the negative section B of $\mathbb{F}_w$, we have

$$(K_{\mathbb{F}_w} + C_1)B = (K_{\tilde{X}} + \tilde{C})B' = \pi^\star K_X \cdot B' = 0.$$

So $K_{\mathbb{F}_w} + C_1 \sim \lambda(B + wA)$, where A is a fibre of $\mathbb{F}_w/\mathbb{P}^1$. Here $\lambda = -2 + m$ where m is the multiplicity of B' in $\tilde{C}$ and $0 \le m \le 1$ since X is log canonical. Hence

$$9 < (K_{\mathbb{F}_w} + C_1)^2 = \lambda^2(B + wA)^2 = \lambda^2 w \le 4w,$$

so w is greater than 2 as required.

Thus $F = B'$ and there are 3 degenerate fibres f_1, f_2, and f_3 of the second form $P \cup \Gamma \cup Q$, where $Q = G^1$, G^2 and G^3 respectively. Recall that G^i is either a single smooth curve of self-intersection $-r_i$ or a chain of (-2)-curves of length $(r_i - 1)$. If the fibre f_i is a chain, we deduce that P is either a chain of (-2)-curves of length $(r_i - 1)$ or a single smooth curve of self-intersection $-r_i$ respectively. Since P contracts to a singularity of type $\frac{1}{n^2}(1, na - 1)$, it follows that $r_i = 4$ and P is a (-4)-curve. If f_i is not a chain, we find that Q is a chain of three (-2)-curves, the (-1)-curve Γ meets the middle component of Q and P is empty, hence again $r_i = 4$. So $r_i = 4$ for each i, contradicting the description of E above. □

Remark 8.6. Conversely, it is well-known that an elliptic cone of degree 9 admits a smoothing to $\mathbb{P}^2$ [Pi].

9 The type B surfaces

We give necessary and sufficient conditions for a surface of type B to admit a smoothing to the plane. Together with the description of the Manetti surfaces in Section 8, this completes the finer classification of the surfaces appearing in stable pairs of degree d not a multiple of 3.

Theorem 9.1. *Let X be a surface of type B. Then X admits a $\mathbb{Q}$-Gorenstein smoothing to $\mathbb{P}^2$ if and only if the following conditions are satisfied:*

(1) The surface X has singularities of the following types:

(a) $\frac{1}{n^2}(1, na - 1)$ where $(a, n) = 1$.
(b) $(xy = 0) \subset \frac{1}{r}(1, -1, a)$ where $(a, r) = 1$.

Moreover there are at most 2 singularities of type (b) of index r greater than 1.

(2) $K_X^2 = 9$

(3) Either (i) $\rho(X_1) = \rho(X_2) = 1$ or (ii) $\{\rho(X_1), \rho(X_2)\} = \{1, 2\}$.

Moreover, in this case, X has unobstructed $\mathbb{Q}$-Gorenstein deformations.

Proof. We first prove that the conditions are necessary. The surface X has only singularities of types (a) and (b) by Propositions 6.1 and 6.2. There are at most two singularities of type (b) by Lemma 9.5. We have $K_X^2 = 9$ since X admits a $\mathbb{Q}$-Gorenstein smoothing to $\mathbb{P}^2$. Finally the Picard numbers of the components of X are as described in (3) by Proposition 6.3.

Now suppose that X satisfies the conditions above. We use the $\mathbb{Q}$-Gorenstein deformation theory developed in Section 3 to prove the existence of a smoothing. We first show that the $\mathbb{Q}$-Gorenstein deformations of X are unobstructed. It is enough to show that $H^p(\mathcal{T}^q_{QG,X}) = 0$ for $p+q=2$. A local canonical covering of X is a hypersurface, hence $H^0(\mathcal{T}^2_{QG,X}) = 0$. The sheaf $\mathcal{T}^1_{QG,X}$ is supported on the singular locus of X, which consists of the double curve Δ together with some isolated points. The curve Δ is smooth and rational; let $i\colon \mathbb{P}^1 \hookrightarrow X$ denote the inclusion of the Δ. Near Δ the sheaf $\mathcal{T}^1_{QG,X}$ equals either $i_\star\mathcal{O}_{\mathbb{P}^1}(1)$ or $i_\star\mathcal{O}_{\mathbb{P}^1}$ by Lemma 9.2 and Lemma 9.3, where the two cases correspond to cases (i) and (ii) of condition (3) respectively. Hence in particular $H^1(\mathcal{T}^1_{QG,X}) = 0$. Finally $H^2(\mathcal{T}^0_{QG,X}) = H^2(\mathcal{T}_X) = 0$ by Lemma 9.4.

We now construct a smoothing of X; we first construct an appropriate first order deformation of X and then extend it to obtain a smoothing. Let $P \in X$ be a point of type $\frac{1}{n^2}(1, na-1)$, then

$$P \in X \cong (xy - z^n = 0) \subset \frac{1}{n}(1,-1,a).$$

Locally at P, the sheaf $\mathcal{T}^1_{QG,X}$ equals the skyscraper sheaf $k(P)$ and a non-zero section corresponds to a first order deformation of the form

$$(xy - z^n + t = 0) \subset \frac{1}{n}(1,-1,a) \times \operatorname{Spec}(k[t]/(t^2)).$$

At the double curve $\Delta \cong \mathbb{P}^1 \overset{i}{\hookrightarrow} X$, the sheaf $\mathcal{T}^1_{QG,X}$ equals either $i_\star\mathcal{O}_{\mathbb{P}^1}$ or $i_\star\mathcal{O}_{\mathbb{P}^1}(1)$. Hence we may pick a section s of $\mathcal{T}^1_{QG,X}$ which is either nowhere zero on Δ or has a unique zero at $Q \in \Delta$, where $Q \in X$ is a normal crossing point. The section s corresponds to a first order deformation of a neighbourhood of Δ in X which is locally of the form

$$(xy + t = 0) \subset \frac{1}{r}(1,-1,a) \times \operatorname{Spec}(k[t]/(t^2))$$

away from the zeroes of s and of the form

$$(xy + zt = 0) \subset \mathbb{A}^3 \times \operatorname{Spec}(k[t]/(t^2))$$

at a zero. Since $H^2(\mathcal{T}_{QG,X}) = 0$, we can lift a section $s \in H^0(\mathcal{T}^1_{QG,X})$ to an element of $T^1_{QG,X}$, so there is a global first order infinitesimal deformation of X which is locally of the forms described above. Given such a first order deformation of X, we can extend it to a $\mathbb{Q}$-Gorenstein deformation $\mathcal{X}/T$ over the germ of a curve since $\mathbb{Q}$-Gorenstein deformations of X are unobstructed. Then the general fibre of $\mathcal{X}/T$ is a smooth del Pezzo surface such that $K^2 = 9$, hence is isomorphic to $\mathbb{P}^2$. $\square$

Lemma 9.2. *([Has], Proposition 3.6) Let X be a surface with two normal irreducible components meeting in a smooth curve Δ. Suppose that X has only singularities of the form $(xy = 0) \subset \frac{1}{r}(1, -1, a)$ at Δ. Then, in a neighbourhood of Δ,*

$$\mathcal{T}^1_{QG,X} \cong \mathcal{O}_\Delta(\Delta_1|_\Delta + \Delta_2|_\Delta).$$

Here Δ_i is the restriction of Δ to X_i and we calculate $\Delta_i|_\Delta$ by moving Δ_i on X_i and restricting to Δ; thus $\Delta_i|_\Delta$ is a $\mathbb{Q}$-divisor on Δ which is well defined modulo linear equivalence. The sum $\Delta_1|_\Delta + \Delta_2|_\Delta$ is a $\mathbb{Z}$-divisor on Δ. In particular, the sheaf $\mathcal{T}^1_{QG,X}$ is a line bundle on Δ of degree $\Delta_1^2 + \Delta_2^2$.

Lemma 9.3. *Let X be a surface of type B satisfying the conditions of Theorem 9.1. Then*

$$\Delta_1^2 + \Delta_2^2 = \begin{cases} 1 & \text{if } \rho(X_1) = \rho(X_2) = 1 \\ 0 & \text{if } \{\rho(X_1), \rho(X_2)\} = \{1, 2\} \end{cases}$$

Proof. Let $\tilde{X}_i \to X_i$ be the minimal resolution of the component X_i of X for $i = 1$ and 2. Then

$$K^2_{\tilde{X}_i} + \rho(\tilde{X}_i) = 10$$

for each i by Noether's formula and

$$K^2_{\tilde{X}_1} + K^2_{\tilde{X}_2} + \rho(\tilde{X}_1) + \rho(\tilde{X}_2) = K^2_{X_1} + K^2_{X_2} + \rho(X_1) + \rho(X_2) + 4\sum(1 - \frac{1}{r_j}),$$

where the r_j are the indices of the non-Gorenstein singularities of X at Δ (cf. Theorem 6.5). Thus

$$K^2_{X_1} + K^2_{X_2} = 20 - (\rho(X_1) + \rho(X_2)) - 4\sum(1 - \frac{1}{r_j}).$$

The condition $K_X^2 = 9$ may be rewritten

$$(K_{X_1} + \Delta_1)^2 + (K_{X_2} + \Delta_2)^2 = 9.$$

Finally,

$$K_{X_i}\Delta_i + \Delta_i^2 = -2 + \sum(1 - \frac{1}{r_j})$$

for each i by adjunction. Solving for $\Delta_1^2 + \Delta_2^2$ we obtain $\Delta_1^2 + \Delta_2^2 = 3 - (\rho(X_1) + \rho(X_2))$. ◻

Lemma 9.4. *Suppose X is a surface of type B which satisfies the conditions of Theorem 9.1. Then $H^2(\mathcal{T}_X) = 0$.*

Proof. We have an exact sequence

$$0 \to \mathcal{O}_{X_1}(-\Delta_1) \oplus \mathcal{O}_{X_2}(-\Delta_2) \to \mathcal{O}_X \to \mathcal{O}_\Delta \to 0.$$

Applying the functor $\mathcal{H}om_{\mathcal{O}_X}(\Omega_X, \cdot)$, we obtain the exact sequence

$$0 \to \mathcal{T}_{X_1}(-\Delta_1) \oplus \mathcal{T}_{X_2}(-\Delta_2) \to \mathcal{T}_X \to \mathcal{H}om_{\mathcal{O}_\Delta}(\Omega_X|_\Delta, \mathcal{O}_\Delta).$$

Thus we have an inclusion $\mathcal{T}_{X_1}(-\Delta_1) \oplus \mathcal{T}_{X_2}(-\Delta_2) \hookrightarrow \mathcal{T}_X$ with cokernel supported on Δ. It follows that the map $H^2(\mathcal{T}_{X_1}(-\Delta_1)) \oplus H^2(\mathcal{T}_{X_2}(-\Delta_2)) \to H^2(\mathcal{T}_X)$ is surjective. So it is enough to show that $H^2(\mathcal{T}_{X_i}(-\Delta_i)) = 0$ for $i = 1$ and 2.

Let (Y, C) denote one of the pairs (X_i, Δ_i). By Serre duality,

$$H^2(\mathcal{T}_Y(-C)) \cong \mathrm{Hom}(\mathcal{T}_Y(-C), \mathcal{O}_Y(K_Y))^\vee = \mathrm{Hom}(\mathcal{T}_Y, \mathcal{O}_Y(K_Y + C))^\vee.$$

We claim that $\mathcal{O}_Y(-K_Y - C)$ has a nonzero global section. Assuming this,

$$\mathrm{Hom}(\mathcal{T}_Y, \mathcal{O}_Y(K_Y + C)) \hookrightarrow \mathrm{Hom}(\mathcal{T}_Y, \mathcal{O}_Y) = H^0(\Omega_Y^{\vee\vee}).$$

Now, letting $\pi : \tilde{Y} \to Y$ be the minimal resolution, we have $\Omega_Y^{\vee\vee} = \pi_\star \Omega_{\tilde{Y}}$ since Y has only quotient singularities ([St1], Lemma 1.11). Thus $h^0(\Omega_Y^{\vee\vee}) = h^0(\Omega_{\tilde{Y}}) = h^1(\mathcal{O}_{\tilde{Y}}) = 0$. So $H^2(\mathcal{T}_Y(-C)) = 0$ as required.

It remains to show that $\mathcal{O}_Y(-K_Y - C)$ has a nonzero global section. Consider the exact sequence

$$0 \to \mathcal{O}_Y(-K_Y - 2C) \to \mathcal{O}_Y(-K_Y - C) \to \mathcal{O}_C(-K_Y - C) \to 0.$$

Now $h^1(\mathcal{O}_Y(-K_Y - 2C)) = h^1(\mathcal{O}_Y(2K_Y + 2C)) = 0$ by Serre duality and Kodaira vanishing (recall that Y is log terminal and $-(K_Y + C)$ is ample). So it is enough to show that $\mathcal{O}_C(-K_Y - C)$ has a nonzero global section. A local calculation shows that $\mathcal{O}_C(-K_Y - C) \cong \mathcal{O}_C(-K_C - S)$, where S is the sum of the singular points of Y lying on C. Now C is isomorphic to $\mathbb{P}^1$, and there are at most 2 singular points of Y on C by assumption, thus $\deg(-K_C - S) \geq 0$ and $\mathcal{O}_C(-K_Y - C)$ has a nonzero global section as required. $\square$

Lemma 9.5. *Let X be a surface of type B that admits a $\mathbb{Q}$-Gorenstein smoothing to $\mathbb{P}^2$. Then X has at most two singularities of the form*

$$(xy = 0) \subset \frac{1}{r}(1, -1, a)$$

where the index r is greater than 1.

Proof. Suppose that X is a counter-example and let $\mathcal{X}/T$ be a $\mathbb{Q}$-Gorenstein smoothing of X to $\mathbb{P}^2$. First assume that $\rho(X_1) = 1$ and $\rho(X_2) = 2$. We claim that there is a Mori contraction $f\colon \mathcal{X} \to \mathcal{Y}/T$ with exceptional locus X_1. Assuming this, we deduce that the special fibre Y of $\mathcal{Y}/T$ has a log terminal singularity such that the exceptional locus of the minimal resolution has a 'fork', i.e., there exists an exceptional curve meeting 3 other exceptional curves. But, by Proposition 6.2, the only possible log terminal singularities on Y are cyclic quotient singularities, so the exceptional locus of the minimal resolution is a chain of curves, a contradiction.

We now prove the existence of the contraction f. We have $\Delta_1^2 + \Delta_2^2 = 0$ by Lemma 9.3 and $\rho(X_1) = 1$ by assumption, hence $\Delta_1^2 > 0$ and $\Delta_2^2 < 0$. Thus Δ_2 generates an extremal ray on X_2. It follows that Δ generates an extremal ray on $\mathcal{X}/T$. The divisor $-K_{\mathcal{X}}$ is relatively ample, so in particular $K_{\mathcal{X}}\Delta < 0$ and there is a corresponding contraction $f\colon \mathcal{X} \to \mathcal{Y}/T$. The exceptional locus of f is the divisor X_1 since Δ_1 generates the group $N_1(X_1)$ of 1-cycles on X_1.

Similiarly, if $\rho(X_1) = \rho(X_2) = 1$, then $\mathcal{X}$ is not $\mathbb{Q}$-factorial and there is a $\mathbb{Q}$-factorialisation $\alpha\colon \hat{\mathcal{X}} \to \mathcal{X}/T$, where the special fibre $\hat{X}$ of $\hat{\mathcal{X}}$ has components $\hat{X}_1 \cong X_1$ and $\hat{X}_2$, a blowup of X_2. Then there is a Mori contraction $f\colon \hat{\mathcal{X}} \to \mathcal{Y}/T$ with exceptional locus $\hat{X}_1$; we obtain a contradiction as above. We construct the $\mathbb{Q}$-factorialisation α explicitly below. Let $P \in X$ be a point at which $\mathcal{X}$ is not $\mathbb{Q}$-factorial, then necessarily $P \in \Delta$ and, working locally analytically at $P \in \mathcal{X}/T$, the family $\mathcal{X}/T$ is of the form

$$(xy + t^k g(z^r, t) = 0) \subset \frac{1}{r}(1, -1, a, 0),$$

where t is a local parameter at $0 \in T$ and $g(z^r, t) \in m_{\mathcal{X},P}$, $t \not| g(z^r, t)$. Let $X_1 = (x = t = 0)$ and $X_2 = (y = t = 0)$. If $r = 1$, let $\alpha\colon \hat{\mathcal{X}} \to \mathcal{X}$ be the blowup of $(x = g = 0) \subset \mathcal{X}$. Then, writing $u = g/x$ and $v = x/g$, the 3-fold $\hat{X}$ has the following affine pieces:

$$(vy + t^k = 0) \subset \mathbb{A}^4_{v,y,z,t}$$

$$(xu = g(z^r, t)) \subset \mathbb{A}^4_{x,u,z,t}$$

Thus $\hat{X}_1$ is isomorphic to X_1 and the morphism $\hat{X}_2 \to X_2$ contracts a smooth rational curve to the point $P \in X_2$. If $r > 1$, we obtain α as the quotient of the above construction applied to the canonical covering of $\mathcal{X}$. Finally $\hat{\mathcal{X}}$ is $\mathbb{Q}$-factorial since $\rho(\hat{X}_1) = 1$ and $\rho(\hat{X}_2) = 2$, cf. Proposition 6.3. $\square$

10 The singularities of D and the relation to GIT

If (X, D) is a stable pair then the pair $(X, (\frac{3}{d} + \epsilon)D)$ is slc for $0 < \epsilon \ll 1$. We show that, in the case $X = \mathbb{P}^2$, this condition is a natural strengthening of the GIT stability condition. Roughly speaking, it is the weakest local analytic condition on D which contains the GIT stability condition. This statement is made precise in Propositions 10.2 and 10.4.

Definition 10.1. Let $P \in X$ be the germ of a smooth surface and D a divisor on X. Suppose given a choice of coordinates x, y at $P \in X$ and weights $(m, n) \in \mathbb{N}^2$. Write $D = (f(x, y) = 0)$ and $f(x, y) = \sum a_{ij} x^i y^j$. The *weight* $\mathrm{wt}(D)$ of D is given by

$$\mathrm{wt}(D) = \min\{mi + nj \mid a_{ij} \neq 0\}.$$

Proposition 10.2. *Let D be a plane curve of degree d. Then $(\mathbb{P}^2, D)$ is a stable pair if and only if for every point $P \in \mathbb{P}^2$, choice of analytic coordinates x, y at P and weights (m, n), we have*

$$\mathrm{wt}(D) < \frac{d}{3}(m + n).$$

Proof. Given a smooth surface X and B a $\mathbb{Q}$-divisor on X, to verify that (X, B) is log canonical it is sufficient to check that, for each weighted blowup

$$f \colon E \subset Y \to P \in X$$

of a point $P \in X$, we have $a(E, X, B) \geq -1$. Here $a = a(E, X, B)$ is the discrepancy defined by the equation

$$K_Y + B' = f^\star(K_X + B) + aE.$$

Putting $X = \mathbb{P}^2$ and $B = (\frac{3}{d} + \epsilon)D$ yields the criterion above. $\square$

Definition 10.3. We say that coordinates x, y at a point $P \in \mathbb{P}^2$ are *linear* if there is a choice of homogeneous coordinates X_0, X_1, X_2 on $\mathbb{P}^2$ such that $x = X_1/X_0$ and $y = X_2/X_0$.

Proposition 10.4. *Let D be a plane curve of degree d. Then $D \hookrightarrow \mathbb{P}^2$ is GIT stable if and only if for every point $P \in \mathbb{P}^2$, choice of linear coordinates x, y at P and weights (m, n), we have*

$$\mathrm{wt}(D) < \frac{d}{3}(m + n).$$

Proof. This is the usual numerical criterion for GIT stability [Mu], restated in a form analogous to Proposition 10.2. ☐

Example 10.5. We give an example of a curve $D \subset \mathbb{P}^2$ such that D is GIT stable but $(\mathbb{P}^2, D)$ is not a stable pair. The curve D is a quintic curve with a singularity $P \in D$ of type $(y^2 + x^{13} = 0) \subset \mathbb{C}^2$. To prove the existence of such a curve, pick analytic coordinates x, y at $P = (1 : 0 : 0) \in \mathbb{P}^2$. Let F be a homogenenous polynomial of degree 5, and write F/X_0^5 as a power series $f(x, y)$ in x and y. Quintics depend on 20 parameters, hence we may choose F so that the coefficients of $1, x, \ldots, x^{12}, y, xy, \ldots, x^6y$ in f vanish. Then, for sufficiently generic choice of x and y, $f(x, y) = \alpha y^2 + \beta x^{13} + \cdots$, where $\alpha \neq 0$, $\beta \neq 0$ and $\cdots$ denotes terms of higher weight with respect to the weights $(2, 13)$ of x and y. In this case, the quintic curve $D = (F = 0)$ has a singularity of the desired type at P. Then $(\mathbb{P}^2, D)$ is not a stable pair by Proposition 10.2 — with respect to the weighting $(2, 13)$ of x and y we have $\mathrm{wt}(f) = 26 > \frac{5}{3}(2 + 13) = 25$. On the other hand, let $D' \to D$ be the resolution of the singularity $P \in D$ induced by a $(2, 13)$ weighted blowup of $\mathbb{P}^2$. We compute that $p_a(D') = 0$, hence D' is a smooth rational curve and D has no additional singular points. Thus D is GIT stable by ([Mu], p. 80).

11 Examples

We give the classification of stable pairs of degrees 4 and 5.

Notation 11.1. Given an embedding of a surface Y in a weighted projective space $\mathbb{P}$, we write kH for a general curve in the linear system $|\mathcal{O}_Y(k)|$. For each surface of type B, we use this notation to describe the inverse image of the double curve on each component.

When we list the singularities of the surfaces X we do not mention the normal crossing singularities $(xy = 0) \subset \mathbb{A}^3$. Similiarly, when we list the possible singularities of (X, D), we do not include the cases where X is smooth or normal crossing and the divisor D is normal crossing.

11.1 Degree 4

Surfaces X:

Surface	Double curve	Singularities
$\mathbb{P}^2$		
$\mathbb{P}(1,1,4)$		$\frac{1}{4}(1,1)$
$\mathbb{P}(1,1,2) \cup \mathbb{P}(1,1,2)$	H, H	$(xy=0) \subset \frac{1}{2}(1,1,1)$

Allowed singularities of (X, D):

X	D
$\mathbb{A}^2_{x,y}$	$(y^2+x^3=0)$
$\frac{1}{4}(1,1)$	0
$(xy=0) \subset \frac{1}{2}(1,1,1)$	0

11.2 Degree 5

Surfaces X:

Surface	Double curve	Singularities
$\mathbb{P}^2$		
$\mathbb{P}(1,1,4)$		$\frac{1}{4}(1,1)$
$X_{26} \subset \mathbb{P}(1,2,13,25)$		$\frac{1}{25}(1,4)$
$\mathbb{P}(1,4,25)$		$\frac{1}{4}(1,1)$, $\frac{1}{25}(1,4)$
$\mathbb{P}(1,1,2) \cup \mathbb{P}(1,1,2)$	H, H	$(xy=0) \subset \frac{1}{2}(1,1,1)$
$\mathbb{P}(1,1,5) \cup (X_6 \subset \mathbb{P}(1,2,3,5))$	$H, 2H$	$(xy=0) \subset \frac{1}{5}(1,-1,1)$
$\mathbb{P}(1,1,5) \cup \mathbb{P}(1,4,5)$	$H, 4H$	$\frac{1}{4}(1,1), (xy=0) \subset \frac{1}{5}(1,-1,1)$

Allowed singularities of (X, D):

X	D
$\mathbb{A}^2_{x,y}$	$(y^2 + x^n = 0)$ for $3 \le n \le 9$
$\mathbb{A}^2_{x,y}$	$(x(y^2 + x^n) = 0)$ for $n = 2, 3$
$\frac{1}{4}(1,1)$	$(y^2 + x^n = 0)$ for $n = 2, 6$
$(xy = 0) \subset \frac{1}{2}(1,1,1)$	$(z = 0)$
$\frac{1}{25}(1,4)$	0
$(xy = 0) \subset \frac{1}{5}(1,-1,1)$	0

Note that $X_{26} \subset \mathbb{P}(1,2,13,25)$ is the surface obtained from $\mathbb{P}(1,4,25)$ by smoothing the $\frac{1}{4}(1,1)$ singularity. The smoothing can be realised inside $\mathbb{P}(1,2,13,25)$. To see this, let $k[U,V,W]$ be the homogeneous coordinate ring of $\mathbb{P}(1,4,25)$ and consider the 2nd Veronese subring $k[U,V,W]^{(2)}$. By picking generators for this ring we obtain the embedding

$$\begin{array}{ccc} \mathbb{P}(1,4,25) & \stackrel{\sim}{\longrightarrow} & (XT = Z^2) \subset \mathbb{P}(1,2,13,25) \\ (U,V,W) & \longmapsto & (X,Y,Z,T) = (U^2, V, UW, W^2) \end{array}$$

Then the smoothing of the $\frac{1}{4}(1,1)$ singularity is given by

$$(XT = Z^2 + tT^{13}) \subset \mathbb{P}(1,2,13,25) \times \mathbb{A}^1_t.$$

Similiarly $X_6 \subset \mathbb{P}(1,2,3,5)$ is the surface obtained from $\mathbb{P}(1,4,5)$ by smoothing the $\frac{1}{4}(1,1)$ singularity.

11.3 Sketch of proof

We describe two different ways to establish the classification of stable pairs of degrees 4 and 5 given above. We note immediately that all the surfaces X occurring are either Manetti surfaces or type B surfaces by Theorem 7.1.

11.3.1 The geometric method

We first classify *semistable* pairs of degree d using the classification of Manetti surfaces X (Theorem 8.2) and the bound on the index of the singularities (Theorem 4.5). The possible surfaces X for $d = 4$ are $\mathbb{P}^2$ and

$\mathbb{P}(1,1,4)$, whereas for $d=5$ we have $\mathbb{P}^2$, $\mathbb{P}(1,1,4)$, $X_{26}\subset\mathbb{P}(1,2,13,25)$ and $\mathbb{P}(1,4,25)$.

We now deduce the classification of the stable pairs of degree d using the following result:

Proposition 11.2. *Every stable pair (X,D) of type B has a smoothing $(\mathcal{X},\mathcal{D})/T$ which is obtained from a smoothing $(\mathcal{Y},\mathcal{D}_{\mathcal{Y}})/T$ of a semistable pair by a divisorial extraction, possibly followed by a flopping contraction. Moreover the divisorial extraction $f\colon(\hat{\mathcal{X}},\hat{\mathcal{D}})\to(\mathcal{Y},\mathcal{D}_{\mathcal{Y}})/T$ is crepant in the following sense: $K_{\hat{\mathcal{X}}}+\frac{3}{d}\hat{\mathcal{D}}=f^\star(K_{\mathcal{Y}}+\frac{3}{d}\mathcal{D}_{\mathcal{Y}})$.*

This is a special case of the 'stabilisation process' described in the proof of Theorem 2.12, which produces a smoothing of a stable pair from a smoothing of a semistable pair. The proof uses the explicit construction in the proof of Lemma 9.5. Restricting to the special fibre, we see that the centre $P\in Y\subset\mathcal{Y}$ of the divisorial contraction is a strictly log canonical singularity of the pair $(Y,\frac{3}{d}D_Y)$. If $d=4$ we deduce that $(P\in Y,D)\cong(\mathbb{A}^2,(y^2+x^4=0))$. For $d=5$ there are three possibilities for $(P\in Y,D)$, namely $(\mathbb{A}^2,(y^2+x^{10}=0))$, $(\mathbb{A}^2,(x(y^2+x^4)=0))$ and $(\frac{1}{4}(1,1),(y^2+x^{10}=0))$. The required divisorial extractions $f\colon\hat{\mathcal{X}}\to\mathcal{Y}$ are then determined by [Hac2]. The special fibre $\hat{X}$ of $\hat{\mathcal{X}}$ is $Y'+E$ where Y' is the strict transform of Y and E is the exceptional divisor of f. The map $Y'\to Y$ is a weighted blowup with respect to some analytic coordinates x, y at $P\in Y$ as above. It is important to note that, for example in the case $Y=\mathbb{P}^2$, these coordinates are not necessarily 'linear coordinates' X_1/X_0, X_2/X_0 corresponding to homogeneous coordinates X_0,X_1,X_2 on $\mathbb{P}^2$. Hence the global structure of the rational surface Y' is a little more complicated than one might expect. Finally, if there is a curve Γ on $Y'\subset\hat{\mathcal{X}}$ such that $K_{\hat{\mathcal{X}}}\Gamma=0$ then there is a flopping contraction $\alpha\colon\hat{\mathcal{X}}\to\mathcal{X}$ with exceptional locus Γ; otherwise $\mathcal{X}=\hat{\mathcal{X}}$. Thus either X is obtained from $\hat{X}$ by contracting the curve $\Gamma\subset Y'$ or $X=\hat{X}$.

11.3.2 The combinatorial method

This approach is carried out carefully in [Hac1]. We set up the following notation: given a stable pair (X,D), let (Y,C) be a component of the pair (X^ν,Δ^ν), where X^ν is the normalisation of X and Δ^ν is the inverse image of the double curve of X. Let $\pi\colon\tilde{Y}\to Y$ be the minimal resolution of Y and define an effective $\mathbb{Q}$-divisor $\tilde{C}$ by the equation

$$K_{\tilde{Y}}+\tilde{C}=\pi^\star(K_Y+C).$$

Assuming Y is not isomorphic to $\mathbb{P}^2$, there exists a birational morphism $\tilde{Y} \to \mathbb{F}_w$; fix one such morphism μ with w maximal and let $p\colon \tilde{Y} \to \mathbb{P}^1$ denote the induced ruling.

We first use the bound on the index of the singularities of X (Theorem 4.5) to write down a list of possible singularities of the pair (Y, C). We deduce the possible forms of the connected components of the divisor $\tilde{C}$. We then analyse how these can embed into the surface $\tilde{Y}$ relative to the ruling p (cf. proof of Theorem 8.5). We deduce a list of candidates for the pairs $(\tilde{Y}, \tilde{C})$ and hence for the pairs (Y, C). Finally we glue these components together to obtain the list of surfaces X.

A The relative S_2 condition

Definition A.1. Let $\mathcal{X}/S$ be a flat family of slc surfaces and $\mathcal{F}$ a coherent sheaf on $\mathcal{X}$. We say $\mathcal{F}$ is S_2 *over* S if $\mathcal{F}$ is flat over S and the fibre $\mathcal{F}_s = \mathcal{F} \otimes k(s)$ satisfies Serre's S_2 condition for each $s \in S$. We say $\mathcal{F}$ is *weakly* S_2 *over* S if, for each open subscheme $i\colon \mathcal{U} \hookrightarrow \mathcal{X}$ whose complement has finite fibres, we have $i_\star i^\star \mathcal{F} = \mathcal{F}$.

Remark A.2. The relative S_2 condition is stable under base change, but this is *not* true for the weak relative S_2 condition.

Lemma A.3. *Let $\mathcal{X}/S$ be a flat family of slc surfaces. Let $\mathcal{F}$ be a sheaf on $\mathcal{X}$ which is S_2 over S. Then $\mathcal{F}$ is weakly S_2 over S.*

Example A.4. The sheaf $\mathcal{O}_\mathcal{X}$ is S_2 over S, hence $i_\star \mathcal{O}_\mathcal{U} = \mathcal{O}_\mathcal{X}$ for $i\colon \mathcal{U} \hookrightarrow \mathcal{X}$ as in A.1. Also, the sheaf $\omega_{\mathcal{X}/S}$ is S_2 over S (since $\omega_{\mathcal{X}/S}$ is flat over S and has fibres $\omega_{\mathcal{X}_s}$ which are S_2 by [KM], Corollary 5.69), so $i_\star \omega_{\mathcal{U}/S} = \omega_{\mathcal{X}/S}$.

Proof. Let $i\colon \mathcal{U} \hookrightarrow \mathcal{X}$ be an open subscheme as in A.1 and let $\mathcal{Z}$ denote the complement of $\mathcal{U}$ with its reduced structure. We work locally at a closed point $P \in \mathcal{Z}$, let $P \mapsto s \in S$. The sheaf $\mathcal{F}_s$ is S_2 by assumption, so there is a regular sequence $x_s, y_s \in m_{\mathcal{X}_s,P}$ for $\mathcal{F}_s$ at P. Now $\mathcal{Z}_s \hookrightarrow \mathcal{X}_s$ is a closed subscheme with support P, hence, replacing x_s, y_s by powers x_s^ν, y_s^ν if necessary, we may assume that they lie in the ideal of $\mathcal{Z}_s$. Note that x_s, y_s is still a regular sequence for $\mathcal{F}_s$ by [Mat], Theorem 16.1. Lift x_s, y_s to elements x, y of the ideal of $\mathcal{Z}$, then x, y is a regular sequence for $\mathcal{F}$ at P ([Mat], p. 177, Corollary to Theorem 22.5). Equivalently, we have an exact sequence

$$0 \to \mathcal{F} \overset{(y,-x)}{\to} \mathcal{F} \oplus \mathcal{F} \overset{(x,y)}{\to} \mathcal{F}.$$

Consider the natural map $\mathcal{F} \to i_\star i^\star \mathcal{F}$, write K for the kernel and C for the cokernel. Then K and C have support contained in the set $\mathcal{Z}$, so any given element of K or C is annihilated by some power of the ideal $\mathcal{I}_\mathcal{Z}$ of $\mathcal{Z}$. So, if $K \neq 0$, there exists $0 \neq g \in K$ such that $\mathcal{I}_\mathcal{Z} g = 0$, then $xg = yg = 0$, contradicting the exact sequence above. Similiarly if $C \neq 0$, there exists $g \in i_\star i^\star \mathcal{F} \backslash \mathcal{F}$ such that $\mathcal{I}_\mathcal{Z} g \subset \mathcal{F}$. Again using the exact sequence above, since $(yg, -xg) \mapsto 0$ we obtain $(yg, -xg) = (yg', -xg')$ for some $g' \in \mathcal{F}$; it follows that $g = g'$, a contradiction. Thus $K = C = 0$, so the map $\mathcal{F} \to i_\star i^\star \mathcal{F}$ is an isomorphism as claimed. $\square$

Lemma A.5. *Let $\mathcal{X}/S$ be a flat family of slc surfaces.*

(1) If $\mathcal{F}$ and $\mathcal{G}$ are coherent sheaves on $\mathcal{X}$ and $\mathcal{G}$ is weakly S_2 over S, then $\mathcal{H}om(\mathcal{F}, \mathcal{G})$ is weakly S_2 over S.

(2) If $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is an exact sequence of coherent sheaves on $\mathcal{X}$ and $\mathcal{F}'$ and $\mathcal{F}''$ are weakly S_2 over S, then $\mathcal{F}$ is weakly S_2 over S.

(3) Let $\mathcal{Z}/S$ be a flat family of slc surfaces and $\pi \colon \mathcal{Z} \to \mathcal{X}$ a finite map over S. If $\mathcal{F}$ is a sheaf on $\mathcal{Z}$ which is weakly S_2 over S then $\pi_\star \mathcal{F}$ is weakly S_2 over S

(4) Let $g \colon T \hookrightarrow S$ be a closed subscheme and $g_\mathcal{X} \colon \mathcal{X}_T \hookrightarrow \mathcal{X}$ the corresponding closed subscheme of $\mathcal{X}$. If $\mathcal{F}$ is a sheaf on $\mathcal{X}_T$ which is weakly S_2 over T then $g_{\mathcal{X}\star}\mathcal{F}$ is weakly S_2 over S.

Proof. Let $i \colon \mathcal{U} \hookrightarrow \mathcal{X}/S$ be an open subscheme whose complement has finite fibres. For $\mathcal{F}$ a sheaf on $\mathcal{X}$, let $\alpha_\mathcal{F}$ denote the natural map $\mathcal{F} \to i_\star i^\star \mathcal{F}$; thus $\mathcal{F}$ is weakly S_2 if and only if $\alpha_\mathcal{F}$ is an isomorphism for each $\mathcal{U}$. To prove (1), observe that the map $\alpha_{\mathcal{H}om(\mathcal{F},\mathcal{G})} \colon \theta \mapsto i_\star i^\star \theta$ has inverse $\psi \mapsto \alpha_\mathcal{G}^{-1} \circ \psi \circ \alpha_\mathcal{F}$. For (2), consider the diagram

$$\begin{array}{ccccccccc}
0 & \to & \mathcal{F}' & \to & \mathcal{F} & \to & \mathcal{F}'' & \to & 0 \\
 & & \downarrow \alpha_{\mathcal{F}'} & & \downarrow \alpha_\mathcal{F} & & \downarrow \alpha_{\mathcal{F}''} & & \\
0 & \to & i_\star i^\star \mathcal{F}' & \to & i_\star i^\star \mathcal{F} & \to & i_\star i^\star \mathcal{F}'' & &
\end{array}$$

The rows are exact, and $\alpha_{\mathcal{F}'}$ and $\alpha_{\mathcal{F}''}$ are isomorphisms by assumption, hence $i_\star i^\star \mathcal{F} \to i_\star i^\star \mathcal{F}''$ is surjective and $\alpha_\mathcal{F}$ is an isomorphism, as required. Parts (3) and (4) follow immediately from the definition of the weak S_2 property. $\square$

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Department of Mathematics, University of Michigan,
Ann Arbor, MI 48109, USA
phacking@umich.edu