MODULI OF SURFACES WITH AN ANTI-CANONICAL CYCLE

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Abstract. We prove a global Torelli theorem for pairs (Y, D) where Y is a smooth projective rational surface and $D \in |-K_Y|$ is a cycle of rational curves, as conjectured by Friedman in 1984. In addition, we construct natural universal families for such pairs.

Contents

1. Introduction

We work throughout over the field $k = \mathbb{C}$. We work in the algebraic category unless explicitly stated otherwise.

Definition 1.1. A *Looijenga pair* (Y, D) is a smooth projective surface Y together with a connected singular nodal curve $D \in |-K_Y|$. Note $p_a(D) = 1$ by adjunction, so D is either an irreducible rational curve with a single node, or a cycle of smooth rational curves. We fix an *orientation* of the cycle D, that is, a choice of generator of $H_1(D, \mathbb{Z}) \cong \mathbb{Z}$, and an ordering $D = D_1 + \cdots + D_n$ of the irreducible components of D compatible with the orientation.

By an *isomorphism of Looijenga pairs* (Y^1, D^1), (Y^2, D^2) we mean an isomorphism $f\colon Y^1 \to Y^2$ such that $f(D^1_i) = D^2_i$ for each $i = 1, \ldots, n$ and f is compatible with the orientations of D^1 and D^2. We write $\mathrm{Aut}(Y, D)$ for the group of automorphisms of a Looijenga pair (Y, D) in this sense.

By the birational classification of surfaces, Y in Definition 1.1 is necessarily rational.

Looijenga pairs were introduced in [L81] as natural log analogs of K3 surfaces. Looijenga studied the cases $n \leq 5$ in detail. Here we consider moduli of Looijenga pairs with no restriction on n. We prove the global Torelli Theorem, conjectured by Friedman in [F84], see Theorem 1.8. We construct natural universal families (§5), give a precise description of the moduli stack of Looijenga pairs (Theorem 6.1) and identify the monodromy group (Theorem 5.15).

The motivation for studying Looijenga pairs comes from several directions. Our initial interest arose from the construction of [GHKI]. There we construct a mirror family to any Looijenga pair (Y, D). If the intersection matrix of the components of D is not negative semi-definite, then our construction yields an algebraic family. We call this the *positive* case. In the sequel [GHKII] to that work, we will apply the Torelli theorem to show that in the positive case the mirror family is the universal family of Looijenga pairs constructed here. This has a striking consequence: our construction of the mirror family endows the fibres with a canonical basis of functions. We call elements of this basis theta functions, as a related construction yields theta functions on abelian varieties. Realizing this as the universal family now endows each affine surface $U = Y \setminus D$ in the family with canonical theta functions. Though these include some of the most classical objects in geometry, e.g., (Y, D) could be a cubic surface with a triangle of lines, in which case U is what Cayley called an affine cubic surface, we do not believe this canonical basis has been previously observed, or even conjectured.

A second application of the universal families is given in [GHKIII], where we show that Looijenga pairs are closely related to rank 2 cluster varieties, and realize the Fock-Goncharov fibration of the cluster $\mathcal{X}$-variety (in the rank 2 case) as a natural quotient of our universal families. (See [FG], [FZ] for the definitions of cluster varieties.) In any event, Looijenga pairs appear in a number of other settings, such as the study of degenerations of $K3$ surfaces: the central fibres for maximal degenerations, type III in Kulikov's classification, are normal crossing unions of such pairs.

Looijenga pairs have an elementary construction:

Definition 1.2. Let $(\bar{Y}, \bar{D})$ be a smooth projective toric surface, where $\bar{D} := \bar{Y} \setminus \mathbb{G}_m^2$ is the toric boundary, i.e., the union of toric divisors of Y. Let $\pi : Y \to \bar{Y}$ be the blowup at some number of smooth points (with infinitely near points allowed) of $\bar{D}$. Let $D \subset Y$ be the strict transform of D. Then (Y, D) is a Looijenga pair, and we call $\pi : Y \to \bar{Y}$ a *toric model* for (Y, D).

Essentially all Looijenga pairs arise in this way (i.e., have a toric model). Indeed, define a *simple toric blowup* $(Y', D') \to (Y, D)$ to be the blowup at a node of D, with D' the reduced inverse image of D. A *toric blowup* is a composition of simple toric blowups. Note (Y', D') is again a Looijenga pair, and the log Calabi-Yau is the same, i.e., $Y' \setminus D' = Y \setminus D$. We then have (see [GHKI], Prop. 1.19) the easy fact:

Lemma 1.3. *Given a Looijenga pair (Y, D) there is a toric blowup (Y', D') such that (Y', D') has a toric model.*

For any question we consider, passing to a toric blowup will be at most a notational inconvenience.

To give a precise statement of our results, we first give a number of basic definitions.

Definition 1.4. Let (Y, D) be a Looijenga pair.

(1) A curve $C \subset Y$ is *interior* if no irreducible component of C is contained in D.
(2) An *internal (-2)-curve* means a smooth rational curve of self-intersection -2 disjoint from D.
(3) (Y, D) is *generic* if it has no internal (-2)-curves.

Any Looijenga pair is deformation equivalent to a generic pair, see Proposition 4.1. Note that, by adjunction, any irreducible interior curve with negative self-intersection number is either a (-1)-curve meeting D transversely at a single smooth point, or an internal (-2)-curve. Note also that if (Y, D) is generic and $\pi : Y \to \bar{Y}$ is a toric model, then the blown up points are necessarily distinct (as opposed to infinitely near).

We next consider the notion of periods of Looijenga pairs. We first note (see Lemma 2.1) that the orientation of D determines a canonical identification $\mathbb{G}_m = \mathrm{Pic}^0(D)$, where the latter is the connected component of the identity of $\mathrm{Pic}(D)$.

Definition 1.5. Let

$$D^\perp := \{\alpha \in \mathrm{Pic}(Y) \,|\, \alpha \cdot [D_i] = 0 \text{ for all } i\}.$$

Restriction of line bundles determines a canonical homomorphism

$$\phi_Y : D^\perp \to \mathrm{Pic}^0(D) = \mathbb{G}_m, \quad L \mapsto L|_D. \tag{1.1}$$

The homomorphism $\phi_Y \in T_{D^\perp} := \mathrm{Hom}(D^\perp, \mathbb{G}_m)$ is called the *period point* of Y.

Note $Y \setminus D$ comes with a canonical (up to scaling) nowhere-vanishing 2-form, ω, with simple poles along D. One can show that ϕ_Y is equivalent to the data of periods of ω over cycles in $H_2(Y \setminus D, \mathbb{Z})$, see [F84]. This motivates the term "period."

As well as the notion of periods, we also need the following additional notions to state the Torelli theorem.

Definition 1.6. Let (Y, D) be a Looijenga pair.

(1) The *roots* $\Phi \subset \mathrm{Pic}(Y)$ are those classes in $D^\perp \subset \mathrm{Pic}(Y)$ with square -2 which are realized by an internal (-2)-curve C on a deformation equivalent pair (Y', D'). More precisely, there is a family $(\mathcal{Y}, \mathcal{D})/S$, a path $\gamma\colon [0,1] \to S$, and identifications $(Y, D) = (\mathcal{Y}_{\gamma(0)}, \mathcal{D}_{\gamma(0)})$, $(Y', D') = (\mathcal{Y}_{\gamma(1)}, \mathcal{D}_{\gamma(1)})$, such that the isomorphism
$$H^2(Y', \mathbb{Z}) \to H^2(Y, \mathbb{Z})$$
induced by parallel transport along γ sends $[C]$ to α.

(2) Let $\Delta_Y \subset \mathrm{Pic}(Y)$ be the set of classes of internal (-2)-curves.

(3) Let $\Phi_Y \subset \Phi \subset \mathrm{Pic}(Y)$ be the subset of roots, α, with $\phi_Y(\alpha) = 1$. Note that $\Delta_Y \subset \Phi_Y \subset \Phi$.

(4) Let $W \subset \mathrm{Aut}(\mathrm{Pic}(Y))$ be the subgroup generated by the reflections
$$s_\alpha : \mathrm{Pic}(Y) \to \mathrm{Pic}(Y), \quad \beta \mapsto \beta + \langle \alpha, \beta \rangle \alpha$$
for $\alpha \in \Phi$. Let $W_Y \subset W$ be the subgroup generated by s_α with $\alpha \in \Delta_Y$.

It is clear from the definitions that Φ is invariant under parallel transport, and Δ_Y, Φ_Y, Φ are all invariant under $\mathrm{Aut}(Y, D)$. Further, the sets Φ, Φ_Y, W, W_Y are easily seen to be invariant under toric blowup. Indeed, let $\tau : (Y', D') \to (Y, D)$ be a blow-up of a node of D. Then under pull-back τ^* of divisors, $D^\perp$ is isomorphic to $(D')^\perp$ as lattices.

We will show that $\Phi_Y = W_Y \cdot \Delta_Y$, see Proposition 3.4.

When $n \leq 5$ and the intersection matrix $(D_i \cdot D_j)$ is negative semidefinite, Φ contains a natural *root basis*, which is central to much of Looijenga's analysis. No such basis exists in general.

Definition 1.7. Let (Y, D) be a Looijenga pair.

(1) The cone $\{x \in \mathrm{Pic}(Y)_\mathbb{R} \mid x^2 > 0\}$ has two connected components. Let C^+ be the connected component containing all the ample classes.

(2) For a given ample H let $\tilde{\mathcal{M}} \subset \mathrm{Pic}(Y)$ be the collection of classes E with $E^2 = K_Y \cdot E = -1$, and $E \cdot H > 0$. Note $\tilde{\mathcal{M}}$ is independent of H, see Lemma 2.13. Let $C^{++} \subset C^+$ be the subcone defined by the inequalities $x \cdot E \geq 0$ for all $E \in \tilde{\mathcal{M}}$.

(3) Let $C^{++}_D \subset C^{++}$ be the subcone where additionally $x \cdot [D_i] \geq 0$ for all i.

By Lemma 2.13, C^+, C^{++}, C^{++}_D and $\tilde{\mathcal{M}}$ are all independent of deformation of Looijenga pairs (i.e., preserved by parallel transport).

Our main result is then:

Theorem 1.8. *(Torelli Theorem) Let $(Y_1, D), (Y_2, D)$ be Looijenga pairs and let*

$$\mu\colon \operatorname{Pic}(Y_1) \to \operatorname{Pic}(Y_2)$$

be an isomorphism of lattices.

Global Torelli: *$\mu = f^*$ for an isomorphism of pairs $f : (Y_2, D) \to (Y_1, D)$ iff all the following hold:*

(1) *$\mu([D_i]) = [D_i]$ for all i.*
(2) *$\mu(C^{++}) = C^{++}$.*
(3) *$\mu(\Delta_{Y_1}) = \Delta_{Y_2}$.*
(4) *$\phi_{Y_2} \circ \mu = \phi_{Y_1}$.*

If f exists, the possibilities are a torsor for $\operatorname{Hom}(N', \mathbb{G}_m)$ *where N' is the cokernel of the map*

$$\operatorname{Pic}(Y) \to \mathbb{Z}^n, \quad L \mapsto (L \cdot D_i)_{1 \le i \le n}.$$

Weak Torelli*: There is an element g in the Weyl group W_{Y_1} such that $\mu \circ g = f^*$ for an isomorphism of pairs $f : (Y_2, D) \to (Y_1, D)$ iff μ satisfies conditions (1),(2), and (4). If g exists, it is unique.*

Remark 1.9. We show that for a Looijenga pair (Y, D) the nef cone $\operatorname{Nef}(Y)$ is the subcone of $\overline{C_D^{++}}$ defined by $x \cdot \alpha \ge 0$ for all $\alpha \in \Delta_Y$. See Lemma 2.15. Thus the global Torelli theorem can be restated as follows: Given Looijenga pairs (Y_1, D) and (Y_2, D) and an isomorphism of lattices $\mu\colon \operatorname{Pic}(Y_1) \to \operatorname{Pic}(Y_2)$, there is an isomorphism $f\colon (Y_2, D) \to (Y_1, D)$ of Looijenga pairs such that $\mu = f^*$ iff $\mu(\operatorname{Nef}(Y_1)) = \operatorname{Nef}(Y_2)$ and $\mu([D_i]) = [D_i]$ for each i.

Remark 1.10. In a preliminary version of this note we claimed the Torelli theorem with (2) replaced by the conditions $\mu(C^+) = C^+$ and $\mu(\Phi) = \Phi$. R. Friedman showed us counterexamples to this statement [F13]. We note the weaker condition $\mu(C^+) = C^+$ is sufficient if D supports a divisor of positive square, or if $\mu(H)$ is ample for some ample H, as either condition is easily seen to imply $\tilde{\mathcal{M}}$, and thus C^{++}, is preserved. In [F13] Friedman gives various sufficient conditions under which (2) may be replaced by the conditions $\mu(C^+) = C^+$ and $\mu(\Phi) = \Phi$ (all have the flavor of guaranteeing that Φ is sufficiently big).

The proof of the global Torelli theorem is carried out in §2. The key point there is the notion of a marked Looijenga pair and periods for marked Looijenga pairs.

Definition 1.11. Let (Y, D) be a Looijenga pair.

(1) A *marking of D* is a choice of points $p_i \in D_i^o$ for each i, where D_i^o denotes the intersection of D_i with the smooth locus of D. This is equivalent to the choice

of an isomorphism $i : D^{\mathrm{can}} \to D$ of D with a fixed cycle of rational curves D^{can}. The possible markings of D are a torsor for $\mathrm{Aut}^0(D) = \mathbb{G}_m^n$, the connected component of the identity of $\mathrm{Aut}(D)$.

(2) Fix (Y_0, D) generic. A *marking of* $\mathrm{Pic}(Y)$ is an isomorphism of lattices $\mu\colon \mathrm{Pic}(Y_0) \to \mathrm{Pic}(Y)$ such that $\mu([D_i]) = [D_i]$ for each i and $\mu(C^{++}) = C^{++}$.

(3) Markings p_i, μ determine a *marked period point*:

$$\phi_{((Y,D),p_i,\mu)} \in T_{Y_0} := \mathrm{Hom}(\mathrm{Pic}(Y_0), \mathbb{G}_m)$$

by

$$\phi(L) := (\mu(L)|_D)^{-1} \otimes \mathcal{O}_D\big(\sum (L \cdot D_i) p_i\big) \in \mathrm{Pic}^0(D) = \mathbb{G}_m. \tag{1.2}$$

The global Torelli theorem is proved by first showing that given a toric model for (Y, D), the marked period point determines the location of the blowups, and hence determines Y: this is essentially the content of Proposition 2.9. A bit more work leads to the global Torelli theorem.

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2. The global Torelli Theorem

Lemma 2.1. *Let D be a cycle of n rational curves, with cyclic ordering of the components. This cyclic ordering induces:*

(1) *An identification $\mathrm{Pic}^0(D) = \mathbb{G}_m$, where the former is the group of numerically trivial line bundles.*

(2) *An identification $\mathrm{Aut}^0(D) = (\mathbb{G}_m)^n$, where the former is the identity component of the automorphism group of D.*

Proof. For (1), the fact that there is an abstract isomorphism $\mathbb{G}_m \cong \mathrm{Pic}^0(D)$ is well-known, and the automorphism group of $\mathbb{G}_m$ as a group is $\{1, -1\}$, so there are only two choices of identification. Here is an explicit construction of an identification determined by the orientation, which will be used throughout. We assume $n \geq 3$, leaving the straightforward modifications for $n = 1, 2$ to the reader. For $L \in \mathrm{Pic}^0(D)$, there is a nowhere-vanishing section $\sigma_i \in \Gamma(L|_{D_i})$. Let $\lambda_i := \sigma_{i+1}(p_{i,i+1})/\sigma_i(p_{i,i+1}) \in \mathbb{G}_m$, where

$p_{i,i+1} := D_i \cap D_{i+1}$. Obviously $\lambda(L) := \prod_i \lambda_i$ is independent of the choice of σ_i. The map $L \mapsto \lambda(L)$ gives the canonical isomorphism.

For (2), let (x_i, y_i) be the homogeneous coordinates on D_i with $x_i = 0$ being the point $D_{i-1} \cap D_i$. Then we take the i^{th} copy of $\mathbb{G}_m$ to act on D_i by $(x_i, y_i) \mapsto (x_i, \lambda y_i)$ for $\lambda \in \mathbb{G}_m$. The i^{th} copy of $\mathbb{G}_m$ acts trivially on D_j for $j \neq i$. $\square$

Recall from Definition 1.2 the notion of a toric model of a Looijenga pair.

Definition 2.2. An *exceptional configuration* for generic (Y, D) means an ordered collection $E_{ij} \in \operatorname{Pic}(Y)$ of classes of exceptional divisors for a toric model. (Here for each i the E_{ij} are the exceptional divisors meeting the component D_i of D.) This is an ordered collection of disjoint interior (-1)-curves. If (Y, D) is not necessarily generic, then by a *limiting configuration* in $\operatorname{Pic}(Y)$ we mean the parallel transport (for the Gauss-Manin connection in a family of Looijenga pairs) of an exceptional configuration on a generic pair.

We say that two exceptional configurations $\{E_{ij}\}, \{F_{ij}\}$ for (Y, D) have the same *combinatorial type* if for each i the number of divisors meeting D_i is the same.

More generally, we extend the notion of exceptional or limiting configuration to mean the data of a toric blowup $(Y', D') \to (Y, D)$ together with an exceptional or limiting configuration on (Y', D').

For generic pairs, limiting and exceptional are the same, see Lemma 4.6.

Definition 2.3. A toric model $\pi : (Y, D) \to (\bar{Y}, \bar{D})$ is an iterated blowup at some collection of (not necessarily distinct) points $q_{ij} \in \bar{D}_i^o$ (where $\bar{D}_i^o \cong \mathbb{G}_m$ is the complement of the nodes of $\bar{D}$ along $\bar{D}_i$). As such, the connected components of the exceptional locus are disjoint unions of chains $E_1 + \cdots + E_r$ of smooth rational curves with self-intersections $-2, -2, \ldots, -1$ (or just a single (-1)-curve), where the length, r, is the number of times we blow up at the corresponding point. This chain supports a unique collection of r reduced connected chains, $C_1, \ldots, C_r$, each of self-intersection -1, ordered by inclusion,

$$C_1 = E_r, C_2 = E_r + E_{r-1}, \quad \ldots \quad C_r = E_r + E_{r-1} + \cdots + E_1.$$

Following Looijenga, we refer to these chains as the *exceptional curves* for this toric model. Each such curve is determined by its class, and they are partially ordered by inclusion. Note if we produce a family $(\mathcal{Y}, \mathcal{D})/S$ of Looijenga pairs by varying the points q_{ij} and choosing an order with which to make the iterated blowups, so that in the general fibre we blow up distinct points, then each of these exceptional curves on Y is the limit of a unique smooth exceptional (-1)-curve on the general fibre.

Remark 2.4. Note that the isomorphism class of a toric Looijenga pair $(\bar{Y}, \bar{D})$ is determined by the intersection numbers $\bar{D}_i^2$. Indeed, the isomorphism type of a smooth projective toric surface is determined by the self-intersection numbers of the components of the boundary divisor (because these determine the fan of the surface, see e.g. [Fu93], §2.5).

Note (Y, D) together with the classes $\{E_{ij}\}$ of exceptional curves do not determine by themselves the points $q_{ij} \in \bar{Y}$. Indeed, the classes determine a birational contraction $p : (Y, D) \to (W, D)$, and (W, D) is abstractly isomorphic to $(\bar{Y}, D)$, but further data is needed to specify an identification: this is the data of a marking of D. In the next couple of lemmas we show that the positions of the q_{ij} are determined by the marked period point. From this the global Torelli result contained in Theorem 1.8 will follow.

Lemma 2.5. *Let (Y, D) be a Looijenga pair. For $\alpha \in \mathrm{Aut}^0(D)$ and $L \in \mathrm{Pic}(D)$ let*

$$\psi_\alpha(L) = L^{-1} \otimes \alpha^*(L) \in \mathrm{Pic}^0(D)$$

This gives a homomorphism $\psi : \mathrm{Aut}^0(D) \to \mathrm{Hom}(\mathrm{Pic}(Y), \mathrm{Pic}^0(D))$ via

$$\psi(\alpha)(L) = \psi_\alpha(L|_D).$$

Under the identifications $\mathrm{Aut}^0(D) = \mathbb{G}_m^n$, $\mathrm{Pic}^0(D) = \mathbb{G}_m$ of Lemma 2.1,

$$\psi(\lambda_1, \ldots, \lambda_n)(L) = \prod_i \lambda_i^{\deg L|_{D_i}}$$

for $L \in \mathrm{Pic}(D)$.

Proof. It's enough to compute $\psi(1, \ldots, 1, \lambda, 1, \ldots, 1)(\mathcal{O}_D(q))$ for $q \in D_j^o$, where λ is in the i^{th} place. Clearly this is $\lambda^{\delta_{ij}}$, as required. □

Proposition 2.6. *There is a long exact sequence*

$$1 \to \ker[\mathrm{Aut}(Y, D) \to \mathrm{Aut}(\mathrm{Pic}(Y))] \to \mathrm{Aut}^0(D) \xrightarrow{\psi} \mathrm{Hom}(\mathrm{Pic}(Y), \mathrm{Pic}^0(D)) \to \mathrm{Hom}(D^\perp, \mathrm{Pic}^0(D)) \to 1$$

where ψ is the map of Lemma 2.5 and the other maps are the canonical restrictions.

Proof. It is easy to see that if $(Y', D') \to (Y, D)$ is a toric blowup, then the result for (Y', D') implies the result for (Y, D), so by Lemma 1.3 we can assume (Y, D) has a toric model $\pi : Y \to \bar{Y}$.

We have the following commutative diagram of exact sequences:

$$
\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & & \\
 & 0 \arrow[r] \arrow[d] & \mathrm{Pic}(\bar{Y}) \arrow[r] \arrow[d, "\pi^*"] & \bigoplus_i \mathbb{Z}\bar{D}_i \arrow[r] \arrow[d, "="] & N \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & D^{\perp} \arrow[r] \arrow[d] & \mathrm{Pic}(Y) \arrow[r] \arrow[d] & \bigoplus_i \mathbb{Z}D_i \arrow[r] \arrow[d] & N' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \bigoplus \mathbb{Z}E_{ij} \arrow[r, "="] & \bigoplus \mathbb{Z}E_{ij} \arrow[r] \arrow[d] & 0 \arrow[r] & 0 & \\
 & & 0 & & &
\end{tikzcd}
$$

Here N is dual to the character lattice, M, of the structure torus of $\bar{Y}$. The first row is the standard description of $A_1(\bar{Y})$, identified with $\mathrm{Pic}(\bar{Y})$ by Poincaré duality, with the map from $\mathrm{Pic}(\bar{Y})$ given by $C \mapsto \sum_i (C \cdot \bar{D}_i)\bar{D}_i$. The map to N takes $\bar{D}_i$ to the first lattice point v_i along the ray of the fan corresponding to $\bar{D}_i$. This exact sequence is the dual of the standard exact sequence describing $\mathrm{Pic}(\bar{Y})$, see e.g., [Fu93], §3.4. The E_{ij}'s are the exceptional curves of π. The map $\mathrm{Pic}(Y) \to \bigoplus_i \mathbb{Z}D_i$ is similarly given by $C \mapsto \sum_i (C \cdot D_i)D_i$.

The kernel of $N \to N'$ is easily seen to be the subgroup $S \subset N$ generated by the rays in the fan for $\bar{Y}$ corresponding to boundary divisors $\bar{D}_i$ along which π is not an isomorphism.

Note that $N = \mathrm{Hom}(N, \bigwedge^2 N)$ via $n \mapsto (n' \mapsto n' \wedge n)$ and the orientation gives a trivialization $\bigwedge^2 N = \mathbb{Z}$, thus an identification $N = M$. Thus

$$\mathrm{Hom}(N/S, \mathbb{G}_m) \subset \mathrm{Hom}(N, \mathbb{G}_m) = \mathrm{Hom}(M, \mathbb{G}_m)$$

is the subgroup of homomorphisms to $\mathbb{G}_m$ whose restriction to S is trivial. Equivalently, these are the automorphisms in $\mathrm{Aut}(\bar{Y}, \bar{D}) = \mathrm{Hom}(M, \mathbb{G}_m)$ fixing pointwise those $\bar{D}_i$ along which π is not an isomorphism. It's easy to see this is identified with

$$\ker\big(\mathrm{Aut}(Y, D) \to \mathrm{Aut}(\mathrm{Pic}(Y))\big).$$

The result follows by applying $\mathrm{Hom}(\cdot, \mathrm{Pic}^0(D))$ to the row of the above commutative diagram describing $\mathrm{Pic}(Y)$. The fact that the middle map coincides with ψ then follows from Lemma 2.5. $\square$

We next show that for a toric Looijenga pair, any possible marked period point can be realised by a particular choice of marking of D.

Lemma 2.7. *Let $(\bar{Y}, D = D_1 + \cdots + D_n)$ be a toric Looijenga pair, including an identification of the torus T acting on $\bar{Y}$ with its open orbit. Let $\bar{\phi} \in \mathrm{Hom}(\mathrm{Pic}(\bar{Y}), \mathrm{Pic}^0(D))$. Then there are points $p_i \in D_i^o \subset \bar{Y}$ such that for any $L \in \mathrm{Pic}(\bar{Y})$,*

$$\bar{\phi}(L) = (L|_D)^{-1} \otimes \bigotimes_{i=1}^{n} \mathcal{O}_D((L \cdot D_i)p_i).$$

Moreover, T acts simply transitively on the possible collections of p_i.

Proof. Start with an arbitrary choice of $p_i \in D_i^o$. The exact sequence of Proposition 2.6 reduces to

$$1 \longrightarrow T \longrightarrow \mathrm{Aut}^0(D) \xrightarrow{\psi} \mathrm{Hom}(\mathrm{Pic}(\bar{Y}), \mathrm{Pic}^0(D)) \longrightarrow 1.$$

Denote the map $L \mapsto (L|_D)^{-1} \otimes \bigotimes_{i=1}^n \mathcal{O}_D((L \cdot D_i)p_i)$ by $\bar{\phi}' \in \mathrm{Hom}(\mathrm{Pic}(\bar{Y}), \mathrm{Pic}^0(D))$. Given any $\alpha \in \mathrm{Aut}^0(D)$, using Lemma 2.5, consider the map

$$\begin{aligned} L \mapsto (L|_D)^{-1} &\otimes \bigotimes_{i=1}^{n} \mathcal{O}_D\big((L \cdot D_i)\alpha^{-1}(p_i)\big) \\ &= \bar{\phi}'(L) \otimes \psi_\alpha \left(\bigotimes_{i=1}^{n} \mathcal{O}_D\big((L \cdot D_i)p_i\big) \right) \\ &= \bar{\phi}'(L) \otimes \psi(\alpha)(L). \end{aligned}$$

So this map coincides with $\bar{\phi}' \otimes \psi(\alpha)$. Thus by replacing p_i with $\alpha^{-1}(p_i)$ for some suitable choice of α, we obtain $\bar{\phi} = \bar{\phi}'$. Furthermore, the possible choices of p_i are a torsor for the kernel of ψ. □

Lemma 2.8. *Let $(\bar{Y}, D)$ be as in Lemma 2.7. The structure of $\bar{Y}$ as a toric variety together with the orientation of D gives a canonical identification of D_i^o with $\mathbb{G}_m$. Let $m_i \in D_i^o$ correspond to $-1 \in \mathbb{G}_m$ under this identification. Define*

$$p_i : \mathrm{Aut}^0(D) \to D_i^o, \quad \alpha \mapsto \alpha^{-1}(m_i).$$

(1)
$$\psi(\alpha)(L) = (L|_D)^{-1} \otimes \bigotimes_{i=1}^{n} \mathcal{O}_D\big((L \cdot D_i)p_i(\alpha)\big) \in \mathrm{Pic}^0(D)$$
for all $\alpha \in \mathrm{Aut}^0(D)$ (ψ as in Lemma 2.5).

(2) *Noting ψ is surjective, let $\gamma : \mathrm{Hom}(\mathrm{Pic}(\bar{Y}), \mathrm{Pic}^0(D)) \to \mathrm{Aut}^0(D)$ be a section of ψ. Let $\bar{p}_i : \mathrm{Hom}(\mathrm{Pic}(\bar{Y}), \mathrm{Pic}^0(D)) \to D_i^o$ be the composition $p_i \circ \gamma$. Then for each $\bar{\phi} \in \mathrm{Hom}(\mathrm{Pic}(\bar{Y}), \mathrm{Pic}^0(D))$, the points $\bar{p}_i(\bar{\phi})$ satisfy the conclusion of Lemma 2.7 for $\bar{\phi}$.*

Proof. (1) amounts to showing that

$$L|_D = \bigotimes_{i=1}^{n} \mathcal{O}_D((L \cdot D_i)m_i).$$

It's enough to do this for an ample line bundle, so we can assume $(\bar{Y}, L)$ is the polarized toric surface given by a lattice polygon. In that case take the section of L given by a sum of monomials corresponding to all lattice points on the boundary, with coefficients chosen so that the restriction of the section to $D_i \cong \mathbb{P}^1$ takes the form $(x+y)^{L \cdot D_i}$. Its zero scheme is exactly $\sum (L \cdot D_i)m_i$.

For (2), note

$$\bar{\phi}(L) = \psi(\gamma(\bar{\phi}))(L) = (L|_D)^{-1} \otimes \bigotimes_{i=1}^{n} \mathcal{O}_D\big((L \cdot D_i)\bar{p}_i(\bar{\phi})\big)$$

by (1), as desired. $\square$

The following contains most of the ideas needed for global Torelli, showing that the marked period point determines a marked Looijenga pair.

Proposition 2.9. *Let (Y, D) be a Looijenga pair and $\{E_{ij}\} \subset \operatorname{Pic}(Y)$ the classes of exceptional curves for a toric model of type $(\bar{Y}, D)$. Let $\phi \in \operatorname{Hom}(\operatorname{Pic}(Y), \operatorname{Pic}^0(D))$.*

1. *There is an inclusion $\operatorname{Pic}(\bar{Y}) \subset \operatorname{Pic}(Y)$ given by pullback. Let $\bar{\phi} : \operatorname{Pic}(\bar{Y}) \to \operatorname{Pic}^0(D)$ be the restriction $\phi|_{\operatorname{Pic}(\bar{Y})}$. Let $p_i \in D_i^o \subset \bar{Y}$ be given by $\bar{\phi}$ from Lemma 2.7. There are unique points $q_{ij} \in D_i^o \subset \bar{Y}$ such that*

$$\phi(E_{ij}) = \mathcal{O}_D(q_{ij})^{-1} \otimes \mathcal{O}_D(p_i).$$

   Let (Z, D) be the iterated blowup along the collection of points (possibly with repetitions) $q_{ij} \subset D_i^o \subset \bar{Y}$. There is a unique isomorphism $\mu : \operatorname{Pic}(Y) \to \operatorname{Pic}(Z)$ preserving boundary classes, and sending E_{ij} to the class of the corresponding exceptional curve. Under this identification, ϕ is the marked period point of $((Z, D), p_i, \mu)$, as defined in (1.2).
2. *Suppose there is a marking $r_i \in D_i^o \subset Y$ so that ϕ is the marked period point for $((Y, D), r_i)$. Then μ is induced by a unique isomorphism of Looijenga pairs between (Y, D) and (Z, D) which sends r_i to p_i.*

Proof. (1) is immediate from the construction. So we assume we have the marking $r_i \in D_i^o$ as in (2). By assumption there is a birational map $\pi : Y \to \bar{Y}$ with exceptional curves $\{E_{ij}\}$, and $\pi^* : \operatorname{Pic}(\bar{Y}) \to \operatorname{Pic}(Y)$ is the inclusion of (1). Now by definition of the marked period point, the points $\pi(r_i)$ satisfy the conclusions of Lemma 2.7 for $\bar{\phi}$. Thus by the uniqueness statement in that lemma, we can change π (composing by a translation in the structure torus of $\bar{Y}$) and assume $\pi(r_i) = p_i$. The points $\pi(E_{ij} \cap D_i)$

satisfy the conditions on the q_{ij}, so by uniqueness $\pi(E_{ij} \cap D_i) = q_{ij}$. Thus π is exactly the same iterated blowup as Z, and so clearly (Y, D) and (Z, D), together with the markings of their boundaries, are isomorphic, by an isomorphism inducing μ. This isomorphism is unique by Proposition 2.6. $\square$

Corollary 2.10. *Let $(Y, D), (Y', D)$ be Looijenga pairs (resp. pairs with marked boundary), having toric models of the same combinatorial type. Let ϕ, ϕ' be the period points (resp. the marked period points). Then there is a unique isomorphism of lattices $\mu : \mathrm{Pic}(Y) \to \mathrm{Pic}(Y')$ preserving the boundary classes and the exceptional curves for the toric models. The isomorphism μ is induced by an isomorphism f of Looijenga pairs (resp. pairs with marked boundary) iff $\phi' \circ \mu = \phi$, and in that case the possible f form a torsor for*

$$\ker\big(\mathrm{Aut}(Y, D) \to \mathrm{Aut}(\mathrm{Pic}(Y))\big)$$

(resp. f is unique).

Proof. The marked case is immediate from Proposition 2.9. For the unmarked case, write $\bar{\phi}, \bar{\phi}'$ for the period points defined by (1.1), and assume $\bar{\phi}' \circ \mu = \bar{\phi}$. Choose arbitrary markings of the boundaries of Y, Y', with marked period points ϕ, ϕ'. Now by Proposition 2.6 we can adjust the marking of the boundary of Y so $\phi' \circ \mu = \phi$. The final torsor statement is clear from Proposition 2.6. $\square$

For a Looijenga pair (Y, D), we define the monodromy group as follows. For $(\mathcal{Y}, \mathcal{D})/S$ an analytic family of Looijenga pairs over a connected base S, a base point $s \in S$, an identification $(\mathcal{Y}_s, \mathcal{D}_s) = (Y, D)$, and a path $\gamma\colon [0, 1] \to S$ with $\gamma(0) = \gamma(1) = s$, we obtain a *monodromy transformation* $\rho(\gamma) \in \mathrm{Aut}(\mathrm{Pic}(Y))$ by parallel transport along the loop γ. The *monodromy group* of (Y, D) is the subgroup of $\mathrm{Aut}(\mathrm{Pic}(Y))$ consisting of all monodromy transformations.

Remark 2.11. We show in Theorem 5.15 that the full monodromy group is realized by an analytic family over a smooth base.

Lemma 2.12. *Let (Y, D) be a Looijenga pair, Φ the associated set of roots, and W the Weyl group of Φ. Then W is contained in the monodromy group of (Y, D).*

Proof. Given $\alpha \in \Phi$, by definition there exists a family of Looijenga pairs $(\mathcal{Y}, \mathcal{D})/S$, a path $\gamma\colon [0, 1] \to S$, and an identification $(Y, D) = (\mathcal{Y}_{\gamma(0)}, \mathcal{D}_{\gamma(0)})$, such that the parallel transport of the class $\alpha \in \mathrm{Pic}(Y) = H^2(Y, \mathbb{Z})$ is realized by an internal (-2)-curve C on $(Y', D') := (\mathcal{Y}_{\gamma(1)}, \mathcal{D}_{\gamma(1)})$. Let $(\bar{Y}', \bar{D}')$ denote the contraction of C. Let $(\mathcal{Y}', \mathcal{D}')/(0 \in T)$ and $(\overline{\mathcal{Y}}', \overline{\mathcal{D}}')/(0 \in \bar{T})$ denote the versal deformations of (Y', D') and $(\bar{Y}', \bar{D}')$ respectively. Then $(0 \in T)$ and $(0 \in \bar{T})$ are smooth germs, the locus $H \subset \bar{T}$ of singular fibers is a smooth hypersurface, and there is a finite morphism $T \to \bar{T}$ of degree 2 with branch locus H and a birational proper morphism $\mathcal{Y}' \to \overline{\mathcal{Y}}' \times_{\bar{T}} T$ which restricts to the minimal resolution of each fiber. See [L81], II.2.4. The monodromy of the family around H is given by the Picard-Lefschetz reflection in the class of $[C]$. Now, using the path γ, we deduce that the reflection s_α lies in the monodromy group of (Y, D). □

Lemma 2.13. *Let (Y, D) be a Looijenga pair. Let $E \in \mathrm{Pic}(Y)$ be a class with $E^2 = K_Y \cdot E = -1$. The following are equivalent:*

(1) *$E \cdot H > 0$ for some nef divisor H.*
(2) *E is effective.*

The cones C^{++} and C^{++}_D defined in Definition 1.7 are invariant under parallel transport for deformations of Looijenga pairs, and under the action of W_Y.

Proof. Obviously (2) implies (1). Riemann–Roch gives (1) implies (2).

Given a family of Looijenga pairs over a base scheme S, working locally analytically on S we can choose an ample divisor, H, on the total space and then compute C^{++} on each fibre using the restriction of H. From this deformation invariance is clear. Invariance under W_Y follows from Lemma 2.12. □

Lemma 2.14. *Let (Y, D) be a Looijenga pair. Let $\mathcal{M} \subset \mathrm{Pic}(Y)$ denote the set of classes of (-1)-curves not contained in D.*

(1) *Let $C \subset Y$ be an irreducible curve. Either $C^2 \geq 0$ or $[C] \in \mathrm{Pic}(Y)$ is in the union of $\mathcal{M}$, Δ_Y and $\{[D_i] \,|\, 1 \leq i \leq n\}$.*
(2) *Let $H \in \mathrm{Pic}(Y)$ be an ample class. Then the closure of the Mori cone of curves $\overline{\mathrm{NE}}(Y)$ is the closure of the convex hull of the union of*

$$C^+ := \{x \in \mathrm{Pic}(Y) \otimes_{\mathbb{Z}} \mathbb{R} \mid x^2 > 0, x \cdot H > 0\}$$

together with Δ_Y, $\mathcal{M}$ and $\{[D_i] \,|\, 1 \leq i \leq n\}$. Equivalently, by Lemma 2.13, $\overline{\mathrm{NE}}(Y)$ is the closure of the convex hull of the union of C^+, Δ_Y, $\tilde{\mathcal{M}}$, and $\{[D_i] \,|\, 1 \leq i \leq n\}$, where

$$\tilde{\mathcal{M}} = \{E \in \mathrm{Pic}(Y) \mid E^2 = K_Y \cdot E = -1 \text{ and } E \cdot H > 0\}$$

for some ample divisor H (as in the definition of C^{++}).

Proof. For (1), let $C \subset Y$, $C \not\subset D$, be irreducible. If $C^2 < 0$ then $C \in \Delta_Y \cup \mathcal{M}$ by adjunction.

For (2), note $C^+ \subset \mathrm{NE}(Y)$ by Riemann-Roch and if C is effective with $C^2 \geq 0$, then C is contained in the closure of C^+. The description of the Mori cone then follows from (1). □

Lemma 2.15. *Let (Y,D) be a Looijenga pair and $H \in \mathrm{Pic}(Y)$ an ample class. Then $\mathrm{Nef}(Y) \subset H^2(Y,\mathbb{R})$ is the closure of the subcone of C_D^{++} defined by the inequalities $x \cdot \alpha \geq 0$ for all $\alpha \in \Delta_Y$.*

Proof. Since $\mathrm{Nef}(Y)$ is the dual cone to $\mathrm{NE}(Y)$, this follows immediately from Lemma 2.14, (2). □

Proof of the global Torelli, Theorem 1.8. If $\mu = f^*$ for an isomorphism f then μ obviously satisfies the conditions, and the possibilities for f are a torsor for $\ker(\mathrm{Aut}(Y,D) \to \mathrm{Aut}(\mathrm{Pic}(Y)))$, as in Corollary 2.10. This is identified in the proof of Proposition 2.6 with $\mathrm{Hom}(N', \mathbb{G}_m)$.

Now assuming we have such a μ, we show it is induced by an isomorphism of pairs. We can replace Y_1 by a toric blowup and Y_2 by the corresponding toric blowup, and so by Lemma 1.3 we can assume Y_1 has a toric model. Then $\mu(\mathrm{Nef}(Y_1)) = \mathrm{Nef}(Y_2)$ by Lemma 2.15. Thus the same is true of the Mori cones of curves by duality. Note also that $\mu(K_{Y_1}) = K_{Y_2}$ since D is anti-canonical.

The exceptional locus of a toric model $Y_1 \to \bar{Y}_1$ is a disjoint union of chains of interior smooth rational curves $F_1, \ldots, F_r$ with self-intersection numbers $-2, -2, \ldots, -2, -1$, such that F_j is disjoint from D for $j < r$ and F_r meets D transversely in one point. (Such a chain is the exceptional locus over a point $p \in D$ which is blown up r times.) By assumption $\mu(\Delta_{Y_1}) = \Delta_{Y_2}$, so μ sends internal (-2)-curves to internal (-2)-curves. Also, the class x of a (-1)-curve is characterized by $x^2 = -1$, $x \cdot K = -1$, and x generates an extremal ray of the Mori cone. Thus μ sends interior (-1)-curves to interior (-1)-curves. Also, since μ preserves the intersection product, the curves in Y_2 corresponding to the exceptional locus of $Y_1 \to \bar{Y}_1$ intersect in the same way, that is, they form a disjoint union of chains. Hence there is a birational morphism $(Y_2, D) \to (\bar{Y}_2, \bar{D})$ which contracts these curves, and is given by a sequence of blowups of the same combinatorial type as $(Y_1, D) \to (\bar{Y}_1, \bar{D})$.

We claim that the surface $(\bar{Y}_2, \bar{D})$ is toric. Let (Y,D) be a Looijenga pair, and write $e(X) = \sum (-1)^i \dim H^i(X, \mathbb{R})$ for the Euler number of a topological space X. If $(Y', D') \to (Y, D)$ is a toric blowup then $Y' \setminus D' = Y \setminus D$ so in particular $e(Y' \setminus D') = e(Y \setminus D)$. If $(Y', D') \to (Y, D)$ is a birational morphism of Looijenga pairs given by blowing up a smooth point of D (and defining D' to be the strict transform of D) then $e(Y' \setminus D') = e(Y \setminus D) + 1$. If (Y,D) is toric then $e(Y \setminus D) = e((\mathbb{C}^\times)^2) = 0$. Now it follows from the existence of toric models (Lemma 1.3) that a Looijenga pair (Y,D) satisfies $e(Y \setminus D) \geq 0$ with equality iff (Y,D) is toric. In our situation we have $e(\bar{Y}_1 \setminus \bar{D}) = e(\bar{Y}_2 \setminus \bar{D})$ (because $e(Y_1 \setminus D) = e(Y_2 \setminus D)$ and the toric models $(Y_1, D) \to (\bar{Y}_1, \bar{D})$ and $(Y_2, \bar{D}) \to (\bar{Y}_2, D)$ have the same number of exceptional curves). Thus $(\bar{Y}_1, \bar{D})$ toric implies $(\bar{Y}_2, \bar{D})$ toric.

Next observe that the toric pairs $(\bar{Y}_1, \bar{D})$ and $(\bar{Y}_2, \bar{D})$ are isomorphic. Indeed, the self-intersection numbers $\bar{D}_i^2$ for $\bar{Y}_1$ and $\bar{Y}_2$ coincide because the self-intersection numbers D_i^2 for Y_1 and Y_2 coincide and the toric models $(Y_1, D) \to (\bar{Y}_1, \bar{D})$ and $(Y_2, D) \to (\bar{Y}_2, \bar{D})$ have the same combinatorial type. So $(\bar{Y}_1, \bar{D})$ and $(\bar{Y}_2, \bar{D})$ are isomorphic by Remark 2.4.

Now we may apply Corollary 2.10. □

3. The weak Torelli theorem

The following result is due to R. Friedman.

Theorem 3.1. *(*[F13]*, Theorem 2.14.) The set Φ of roots coincides with the set of classes $\alpha \in \operatorname{Pic}(Y)$ such that $\alpha^2 = -2$, $\alpha \cdot D_i = 0$ for each i, and the associated hyperplane $\alpha^\perp$ meets the interior of C_D^{++}.*

We recall the following statement about the action of Weyl groups:

Theorem 3.2. *The arrangement of hyperplanes*

$$\alpha^\perp \subset C^{++}, \quad \alpha \in W_Y \cdot \Delta_Y$$

is locally finite. The group W_Y acts simply transitively on the Weyl chambers, and each chamber is a fundamental domain for the action of W_Y on C^{++}. One chamber is defined by the inequalities $x \cdot \alpha \geq 0$ for all $\alpha \in \Delta_Y$ (and for each $\alpha \in \Delta_Y$ the equation $x \cdot \alpha = 0$ defines a codimension one face of this chamber). The analogous statements hold for the Weyl chambers of C_D^{++}.

Proof. The analogous statement for chambers in C^+ is a basic result in the theory of hyperbolic reflection groups, see [D08], Theorem 2.1. This immediately implies the result for the chambers in C^{++} or C_D^{++} , as these full dimensional subcones of C^+ are preserved by W_Y, see Lemma 2.13. The closure of the chamber in C_D^{++} defined by $x \cdot \alpha \geq 0$ for each $\alpha \in \Delta_Y$ is identified with the nef cone of Y by Lemma 2.15. By definition the elements of Δ_Y are the classes of (-2)-curves on Y and thus define codimension one faces of the nef cone. □

Lemma 3.3. *Let (Y, D) be a Looijenga pair. Let L be a line bundle on Y such that $L^2 = -2$ and $L|_D \simeq \mathcal{O}_D$. Then $h^0(L) > 0$ or $h^0(L^{-1}) > 0$.*

Proof. Suppose $H^0(L) = 0$. Using the exact sequence

$$0 \to L \otimes \mathcal{O}_Y(-D) \to L \to \mathcal{O}_D \to 0$$

we see that $H^0(L\otimes\mathcal{O}_Y(-D))=0$ and $H^1(L\otimes\mathcal{O}_Y(-D))\neq 0$. Equivalently, by Serre duality, $H^1(L^{-1})\neq 0$ and $H^2(L^{-1})=0$. Now by the Riemann–Roch formula

$$h^0(L^{-1})>\chi(L^{-1})=\chi(\mathcal{O}_Y)+\frac{1}{2}L^{-1}\cdot(L^{-1}-K_Y)=0.$$

□

Proposition 3.4. *Let (Y,D) be a Looijenga pair. Then $\Phi_Y=W_Y\cdot\Delta_Y$.*

Proof. (cf. [F13], Proof of Theorem 2.14). Note that W preserves Φ by Lemma 2.12 and W_Y preserves the period point $\phi_Y\colon D^\perp\to\mathbb{G}_m$. It follows that $W_Y\cdot\Delta_Y\subset\Phi_Y$. Conversely, given $\alpha\in\Phi_Y$, we show $\alpha\in W_Y\cdot\Delta_Y$. By Theorem 3.1 there exists a class x in the interior of C^{++}_D such that $x\cdot\alpha=0$. In particular $x\cdot[D_i]>0$ for each i. We may assume x is an integral class, say $x=[H]$. By Lemma 2.15 and Theorem 3.2, replacing x and α by wx and $w\alpha$ for suitable $w\in W_Y$, we may assume x lies in the nef cone of Y. Also $x^2>0$ (because $\alpha^2=-2<0$ and $x\cdot\alpha=0$). So H is nef and big. By Lemma 3.3, replacing α by $-\alpha$ if necessary, we may assume that α is effective, say $\alpha=\sum a_i[C_i]$ for some irreducible curves $C_i\subset Y$ and $a_i\in\mathbb{N}$. Now $\alpha\cdot H=0$ implies $C_i\cdot H=0$ for each i. In particular no C_i is a component of D, so $\alpha\cdot D=0$ implies $C_i\cdot D=0$ for all i. Also, the span of the classes of the C_i is negative definite. Now by adjunction each C_i is a (-2)-curve, and $\bigcup C_i$ is a configuration of (-2)-curves with dual graph a Dynkin diagram of type A, D, or E. (Note that $\bigcup C_i$ is connected because it is the support of the cycle $\sum a_iC_i$ with square -2.) Finally, the Weyl group of a root system of type A, D, or E acts transitively on the set of roots (and the roots are precisely the elements β of the root lattice such that $\beta^2=-2$). So $\alpha\in W_Y\cdot\Delta_Y$. □

Corollary 3.5. *Let (Y,D) be a Looijenga pair. Then (Y,D) is generic iff $\phi_Y(\alpha)\neq 1$ for all $\alpha\in\Phi$.*

Proof. By definition (Y,D) is generic iff $\Delta_Y=\emptyset$. This is equivalent to $\Phi_Y=\emptyset$ by Proposition 3.4. □

Proof of the weak Torelli Theorem. Note that W_{Y_1} fixes ϕ_{Y_1} and the $[D_i]$ by the definitions, and preserves C^{++} by Lemma 2.13. So the conditions on the isomorphism μ of lattices are necessary. Conversely, suppose given μ satisfying the hypotheses. The isomorphism μ satisfies $\mu(\Phi)=\Phi$ by Theorem 3.1 and hence $\mu(\Phi_{Y_1})=\Phi_{Y_2}$ by condition (4) of the statement of weak Torelli. Also $\Phi_{Y_i}=W_{Y_i}\cdot\Delta_{Y_i}$ for each $i=1,2$ by Proposition 3.4. Thus μ sends the W_{Y_1}-Weyl chambers of $C^{++}_D\subset\operatorname{Pic}(Y_1)_{\mathbb{R}}$ to the W_{Y_2}-Weyl chambers of $C^{++}_D\subset\operatorname{Pic}(Y_2)_{\mathbb{R}}$. Since W_{Y_1} acts simply transitively on the W_{Y_1}-Weyl chambers of C^{++}_D, there exists a unique $g\in W_{Y_1}$ such that $\mu\circ g$ satisfies $\mu(\Delta_{Y_1})=\Delta_{Y_2}$. Now the global Torelli Theorem applies. □

4. First properties of the monodromy group

Proposition 4.1. *Let (Y, D) be a Looijenga pair. Let $(0 \in \mathrm{Def}(Y, D))$ denote the versal deformation space of the pair and $T'_Y = \mathrm{Hom}(D^\perp, \mathbb{G}_m)$.*

(1) *The local period mapping*

$$\phi\colon (0 \in \mathrm{Def}(Y, D)) \to (\phi_Y \in T'_Y)$$

is a local analytic isomorphism.

(2) *The locus of generic pairs in $\mathrm{Def}(Y, D)$ is the complement of the inverse image under ϕ of the countable union of hypertori*

$$T'_\alpha = \{\psi \in T'_Y \mid \psi(\alpha) = 1\}$$

for $\alpha \in \Phi$.

In particular, every Looijenga pair is a deformation of a generic pair.

Proof. The period mapping is a local isomorphism by [L81], II.2.5.
Statement (2) follows from Corollary 3.5. □

Definition 4.2. Let (Y, D) be a Looijenga pair. Let Adm_Y denote the subgroup of automorphisms of the lattice $\mathrm{Pic}(Y)$ preserving the boundary classes $[D_i]$ and the cone C^{++} (see Definition 1.7). We say an automorphism θ of $\mathrm{Pic}(Y)$ is *admissible* if $\theta \in \mathrm{Adm}_Y$.

Lemma 4.3. *Let (Y, D) be a Looijenga pair. The group Adm_Y contains the monodromy group of (Y, D) and preserves Φ.*

Remark 4.4. In fact we show in Theorem 5.15 that Adm_Y is equal to the monodromy group.

Proof. The monodromy group preserves the cone C^{++} by Lemma 2.13, so it is contained in Adm_Y. The group Adm_Y preserves Φ by Theorem 3.1.

□

Lemma 4.5. *Let (Y, D) be a generic Looijenga pair and $\theta\colon \mathrm{Pic}(Y) \to \mathrm{Pic}(Y)$ an isomorphism of lattices such that $\theta([D_i]) = [D_i]$ for each i. The following conditions are equivalent:*

(1) *$\theta \in \mathrm{Adm}_Y$.*
(2) *$\theta(\mathrm{Nef}(Y)) = \mathrm{Nef}(Y)$.*
(3) *There exists $H \in \mathrm{Pic}(Y)$ such that H and $\theta(H)$ are ample.*

Proof. By definition $\theta \in \mathrm{Adm}_Y$ iff $\theta(C^{++}) = C^{++}$, and $\mathrm{Nef}(Y) = \overline{C_D^{++}}$ by Lemma 2.15 because (Y, D) is generic. So (1) implies (2). Clearly (2) implies (3) (because the ample cone is the interior of the nef cone). Finally, suppose θ satisfies (3). Then θ preserves the set $\tilde{\mathcal{M}}$ and hence the cone C^{++} (see Definition 1.7). So (3) implies (1) and the equivalence of the statements is proved. □

Lemma 4.6. *Let (Y, D) be a generic Looijenga pair. Then any limiting configuration on Y is an exceptional configuration.*

Proof. By definition, a limiting configuration on Y is the parallel transport of an exceptional configuration on a generic pair (Y_0, D). Note that $\mathrm{Nef}(Y_0)$ and $\mathrm{Nef}(Y)$ are identified under parallel transport (because for a generic pair the nef cone coincides with $\overline{C_D^{++}}$ by Lemma 2.15, and this cone is invariant under parallel transport by Lemma 2.13). The elements of the exceptional configuration on Y_0 define codimension one faces of $\mathrm{Nef}(Y_0)$. Hence the elements E_{ij} of the limiting configuration define codimension one faces of $\mathrm{Nef}(Y)$. Now by Lemma 2.14(1) and the intersection numbers it follows that the E_{ij} are a collection of disjoint interior (-1)-curves. As in the proof of the global Torelli theorem, contracting these curves yields a toric pair $(\bar{Y}, \bar{D})$, so $\{E_{ij}\}$ is an exceptional configuration. □

Theorem 4.7. *Let (Y, D) be a Looijenga pair. The group Adm_Y acts simply transitively on the set of limiting configurations of (any given) combinatorial type.*

Proof. We may assume that (Y, D) is generic by Proposition 4.1.

We show that if $\pi\colon (Y', D') \to (Y, D)$ is a toric blowup, then we have a natural identification $\mathrm{Adm}_{Y'} = \mathrm{Adm}_Y$. Note that (Y, D) generic implies (Y', D') generic by the definition of generic, so we may use the equivalent conditions above. We may assume that π is a simple toric blowup, with unique exceptional divisor E. Given $\theta \in \mathrm{Adm}_Y$, we define a homomorphism $\theta'\colon \mathrm{Pic}(Y') \to \mathrm{Pic}(Y')$ by $\theta'(\pi^*\alpha) = \pi^*\theta(\alpha)$ and $\theta'([E]) = [E]$. We claim that $\theta' \in \mathrm{Adm}_{Y'}$. It is clear that θ' is an isomorphism of lattices and $\theta'([D'_i]) = [D'_i]$ for each component D'_i of the boundary $D' \subset Y'$. Letting $H \in \mathrm{Pic}(Y)$ be ample, then $\theta(H)$ is also ample on Y. Now for $N \in \mathbb{N}$ sufficiently large, $H' := N\pi^*H - E$ and $\theta'(H')$ are ample on Y'. So $\theta' \in \mathrm{Adm}_{Y'}$. The map $\mathrm{Adm}_Y \to \mathrm{Adm}_{Y'}$ defined in this way is clearly a group homomorphism. Conversely, given $\theta' \in \mathrm{Adm}_{Y'}$, we have $\theta'([E]) = [E]$. Thus we can define $\theta\colon \mathrm{Pic}(Y) \to \mathrm{Pic}(Y)$ by restricting θ' to $E^\perp$ and using the identification $E^\perp = \mathrm{Pic}(Y)$ given by π^*. Then θ is an isomorphism of lattices and $\theta([D_i]) = [D_i]$ for each i. Now letting H' be ample on Y', then $\theta'(H')$ is also ample on Y'. Hence $H := \pi_*H'$ and $\theta(H) = \pi_*(\theta'(H'))$ are ample on Y, so $\theta \in \mathrm{Adm}_Y$. This defines a homomorphism $\mathrm{Adm}_{Y'} \to \mathrm{Adm}_Y$ which is clearly the inverse of the homomorphism described above.

Let $\theta \in \mathrm{Adm}_Y$, and let $\{E_{ij}\}$ be an exceptional configuration on a toric blowup (Y', D') of (Y, D). We show that $\{\theta(E_{ij})\}$ is another exceptional configuration of the same combinatorial type. Using the identification $\mathrm{Adm}_Y = \mathrm{Adm}_{Y'}$ proved above, we may assume $Y = Y'$. We have $\theta(\mathrm{Nef}(Y)) = \mathrm{Nef}(Y)$, so the $\theta(E_{ij})$ define codimension one faces of $\mathrm{Nef}(Y)$. We can now conclude as in the proof of Lemma 4.6 above.

Conversely, let $\{E_{ij}\}$, $\{F_{ij}\}$ be two exceptional configurations on (Y, D) of the same combinatorial type. Clearly there is a unique isomorphism of lattices $\theta\colon \mathrm{Pic}(Y) \to \mathrm{Pic}(Y)$ such that $\theta([D_i]) = [D_i]$ for all i and $\theta([E_{ij}]) = [F_{ij}]$ for all i and j. We must show that $\theta \in \mathrm{Adm}_Y$. Let $\pi\colon (Y, D) \to (\bar{Y}, \bar{D})$ denote the contraction of the $\{E_{ij}\}$, and $\pi'\colon (Y, D) \to (\bar{Y}, \bar{D})$ the contraction of the $\{F_{ij}\}$. (Note that the toric pairs $(\bar{Y}, \bar{D})$ obtained by the contractions are (non-canonically) isomorphic because the exceptional configurations have the same combinatorial type.) Let $\bar{H} = \sum a_i \bar{D}_i$ be ample on $\bar{Y}$. Then for $N \in \mathbb{N}$ sufficiently large both $H = N\pi^*\bar{H} - \sum E_{ij}$ and $\theta(H) = N(\pi')^*\bar{H} - \sum F_{ij}$ are ample on Y. So $\theta \in \mathrm{Adm}_Y$. □

5. Automorphisms, universal families, and the monodromy group

Given (Y_0, D) a generic Looijenga pair, let (Y_e, D) be a Looijenga pair deformation equivalent to (Y_0, D) with period point ϕ_{Y_e} given by $\phi_{Y_e}(\alpha) = 1$ for all $\alpha \in D^\perp \subset \mathrm{Pic}(Y_e)$. (Note that existence of (Y_e, D) follows from the construction of Proposition 2.9, and (Y_e, D) is uniquely determined up to isomorphism by the weak Torelli theorem.)

We analyze the relationship between the Weyl group W, the group Adm_{Y_0}, and the automorphisms groups of Looijenga pairs deformation equivalent to (Y_0, D).

Theorem 5.1. *Let (Y_0, D) be a generic Looijenga pair and define (Y_e, D) as above. Then $W \subset \mathrm{Adm}_{Y_0}$ is a normal subgroup and there is an exact sequence*

$$1 \to \mathrm{Hom}(N', \mathbb{G}_m) \to \mathrm{Aut}(Y_e, D) \to \mathrm{Adm}_{Y_0}/W \to 1 \tag{5.1}$$

where N' is the group defined in Theorem 1.8.

More generally, for (Y, D) an arbitrary Looijenga pair deformation equivalent to (Y_0, D), let $\mathrm{Hodge}_Y \subset \mathrm{Adm}_{Y_0}$ denote the stabilizer of the period point ϕ_Y (for some choice of marking of $\mathrm{Pic}(Y)$). Then we have an exact sequence

$$1 \to \mathrm{Hom}(N', \mathbb{G}_m) \to \mathrm{Aut}(Y, D) \to \mathrm{Hodge}_Y/W_Y \to 1 \tag{5.2}$$

Proof. Note $W_Y \subset \mathrm{Adm}_{Y_0}$ and Adm_{Y_0} preserves Φ by Lemma 2.12 and Lemma 4.3. Now, since

$$\Phi_Y = \{\alpha \in \Phi \mid \phi_Y(\alpha) = 1\},$$

the group Hodge_Y preserves Φ_Y and $W_Y \subset \mathrm{Hodge}_Y$ is normal. The image of $\mathrm{Aut}(Y, D)$ in $\mathrm{Aut}(\mathrm{Pic}(Y_0))$ has trivial intersection with W_Y, since it preserves the Weyl chamber

$\mathrm{Nef}(Y) \subset \overline{C_D^{++}}$, while the Weyl group acts simply transitively on the chambers. Take $g \in \mathrm{Adm}_{Y_0}$. Composing g with an element of W_Y we can assume g preserves the Weyl chamber $\mathrm{Nef}(Y)$, and thus Δ_Y (as each $\alpha \in \Delta_Y$ corresponds to a codimension one face of the chamber). Now g is in the image of $\mathrm{Aut}(Y, D)$ iff it fixes the period point ϕ_Y by the global Torelli Theorem. Thus the homomorphism $\mathrm{Aut}(Y, D) \to \mathrm{Hodge}_Y / W_Y$ is surjective. Now the exactness follows from Proposition 2.6.

Finally, for Y_e the period point equals the identity element of $\mathrm{Hom}(D^\perp, \mathbb{G}_m)$, so $\mathrm{Hodge}_{Y_e} = \mathrm{Adm}_{Y_0}$ and $W_{Y_e} = W$ by Proposition 3.4. □

Remark 5.2. Note that the description of the automorphism groups of certain Looijenga pairs in [L81], Corollary I.5.4 is incorrect as stated. (The assumption that the automorphism acts trivially on $D^\perp$ should be added to the statement. Moreover, the group $\mathbb{Z}/s\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ in the statement should be replaced by the dihedral group of order $2s$.) The group N' is trivial in the cases studied by Looijenga so $\mathrm{Aut}(Y, D) = \mathrm{Hodge}_Y / W_Y$.

Example 5.3. We give an example where Adm_{Y_0} / W is nontrivial (in fact, infinite). Let D be a cycle of seven (-2)-curves. Then one can show that $\mathrm{Aut}(Y_e, D)$ is infinite. (Indeed, since $\mathcal{O}_{Y_e}(D)|_D \simeq \mathcal{O}_D$, there is an elliptic fibration $f\colon Y_e \to \mathbb{P}^1$ with $f^{-1}(\infty) = D$. Moreover, the fibration f is relatively minimal because $K_X = -D$. So there is an action of the Mordell–Weil group $\mathrm{MW}(f)$ of sections of f on (Y_e, D) given by translation by the section on the smooth fibers of f. Finally, $\mathrm{MW}(f)$ is infinite by [MP86], Theorem 4.1 (or a short root theoretic calculation, cf. Example 5.6 below).) The group N' is finite because $[D_1], \ldots, [D_n] \in \mathrm{Pic}(Y_0)$ are linearly independent. Hence Adm_{Y_0} / W is infinite by Theorem 5.1.

We note by way of comparison:

Lemma 5.4. *In the cases Looijenga considers in* [L81] *we have* $\mathrm{Adm}_Y = W$.

Proof. We use [L81], Proposition I.4.7, p. 284. By definition $\mathrm{Cr}(Y, D)$ is the group of automorphisms of the lattice $\mathrm{Pic}(Y)$ preserving the ample cone of Y and the boundary divisors $D_1, \ldots, D_n$. We may assume (Y, D) is generic, that is, in Looijenga's notation $B^n = \emptyset$. Then $\mathrm{Adm}_Y = \mathrm{Cr}(Y, D) = W$ by Lemma 4.5 and [L81], I.4.7. □

Example 5.5. We describe an example of a Looijenga pair (Y_e', D_e') such that the set Δ of internal (-2)-curves on Y_e' is infinite.

Let (Y_e, D_e) be the Looijenga pair of Example 5.3. Then there is an elliptic fibration $f\colon Y_e \to \mathbb{P}^1$ with $D = f^{-1}(\infty)$ and such that f has infinitely many sections. Each section $C \subset Y_e$ is a (-1)-curve (because $-K_{Y_e} \cdot C = D \cdot C = 1$). Any two sections meeting the same component of D intersect D at the same point (because ϕ_{Y_e} is trivial by definition). So there is a point p in the smooth locus of D_e such that there are infinitely many (-1)-curves on Y_e passing through p. Now let (Y', D') denote the blowup of $p \in Y_e$ together with the strict transform of D_e. Then clearly $\phi_{Y'}$ is also trivial, so $(Y', D') = (Y'_e, D'_e)$. Now Y' contains infinitely many internal (-2)-curves given by the strict transforms of the (-1)-curves in Y_e passing through p.

Example 5.6. We describe an example of a Looijenga pair (Y, D) such that W is trivial and Adm is infinite.

Let $(\bar{Y}, \bar{D})$ be the toric Looijenga pair given by $\mathbb{F}_1$ together with its toric boundary. Label the boundary divisors $\bar{D}_1, \ldots, \bar{D}_4$ so that $\bar{D}_1^2 = -1$, $\bar{D}_2^2 = 0$, $\bar{D}_3^2 = 1$, and $\bar{D}_4^2 = 0$. Let $(\bar{Y}', \bar{D}')$ be the toric pair obtained from $(\bar{Y}, \bar{D})$ by the following sequence of toric blowups. We first blowup $\bar{D}_1 \cap \bar{D}_2$, $\bar{D}_2 \cap \bar{D}_3$, and $\bar{D}_3 \cap \bar{D}_4$, then blowup the intersection point of the strict transform of $\bar{D}_4$ with the exceptional divisor over $\bar{D}_3 \cap \bar{D}_4$. Now let (Y, D) be the Looijenga pair given by performing an interior blowup at a point of each of the (-1)-curves contained in $\bar{D}'$. Then D is a cycle of eight (-2)-curves. One can check that $D^\perp$ does not contain any classes α such that $\alpha^2 = -2$. Thus $\Phi = \emptyset$ and W is trivial for (Y, D). Moreover, choosing the positions of the interior blowups appropriately (so that $(Y, D) = (Y_e, D_e)$), there is an elliptic fibration $f \colon Y \to \mathbb{P}^1$ with $f^{-1}(\infty) = D$. There are no reducible fibers of f besides D (because $\Phi = \emptyset$ and f is relatively minimal). It follows that the Mordell-Weil group of f is infinite. (Indeed, writing $\eta \in \mathbb{P}^1$ for the generic point, the Mordell–Weil group of sections of the elliptic fibration f is given by

$$\mathrm{MW}(f) = \mathrm{Pic}^0(Y_\eta) = \langle D \rangle^\perp / \langle \Gamma \mid f_*\Gamma = 0 \rangle$$

$$= \langle D \rangle^\perp / \langle D_1, \ldots, D_8 \rangle.$$

In particular, $\operatorname{rk} \mathrm{MW}(f) = 1$.) Thus the group $\mathrm{Aut}(Y, D)$ is infinite. Now by Theorem 5.1 we find that Adm is infinite.

Recall from Definition 1.11 that if $((Z, D), p_i)$ is a Looijenga pair with marked boundary, and $\mu : \mathrm{Pic}(Y) \to \mathrm{Pic}(Z)$ is a marking of $\mathrm{Pic}(Z)$, the marked period point of $((Z, D), p_i, \mu)$ is a point in

$$T_Y := \mathrm{Hom}(\mathrm{Pic}(Y), \mathrm{Pic}^0(D)).$$

Construction 5.7. *Universal families.* Let (Y, D) be a Looijenga pair, and $\pi : Y \to \bar{Y}$ a toric model, with exceptional divisors $\{E_{ij}\}$ which are disjoint interior (-1)-curves. Varying $\phi \in T_Y$, the construction of Proposition 2.9 produces sections $p_i : T_Y \to T_Y \times D_i^o \subset T_Y \times \bar{Y}$, and then unique sections $q_{ij} : T_Y \to T_Y \times D_i^o$ such that

$$\phi(E_{ij}) = \mathcal{O}_D(q_{ij}(\phi))^{-1} \otimes \mathcal{O}_D(p_i(\phi)) \in \mathrm{Pic}^0(D) = \mathbb{G}_m.$$

Explicitly, let p_i be the section $\bar{p}_i$ of Lemma 2.8 (this involves choosing the right inverse γ of ψ, but see Remark 5.8), then $q_{ij}(\phi) \in \mathbb{G}_m$ is the point

$$\phi(E_{ij})^{-1} \cdot p_i(\phi) \in D_i^o,$$

where $\mathrm{Pic}^0(D) = \mathbb{G}_m$ acts on D_i^o using the convention of Lemma 2.1.

Let $\Pi : (\mathcal{Y}_{\{E_{ij}\}}, \mathcal{D}) \to T_Y \times \bar{Y}$ be the iterated blowup along the sections

$$q_{ij} \subset T_Y \times D_i^o \subset T_Y \times \bar{Y}.$$

This comes with a marking $\mu : \mathrm{Pic}(Y) \to \mathrm{Pic}(\mathcal{Y})$ preserving boundary classes, and sending E_{ij} to the corresponding exceptional divisor $\mathcal{E}_{ij}$. This induces a marking of $\mathrm{Pic}(Z)$ for each fibre Z. We call $\lambda : (\mathcal{Y}_{\{E_{ij}\}}, p_i, \mu) \to T_Y$ a *universal family.* See Theorem 6.1 for justification of this term.

If $\tau : Y \to Y'$ is a toric blowup, with exceptional divisor E, and Y has a toric model as above, then there is a divisorial contraction $\tilde{\tau} : \mathcal{Y}_{\{E_{ij}\}} \to \tilde{\mathcal{Y}}'_{\{E_{ij}\}}$ which blows down the (-1)-curve $\mu(E)$ in each fibre — this is a family of toric blowups. Observe that identifying $\mathrm{Pic}(Y)$, $\mathrm{Pic}(Y')$ with $A_1(Y)$, $A_1(Y')$ respectively, we have a map $\tau_* : A_1(Y) \to A_1(Y')$, and hence a transpose map $T_{Y'} = \mathrm{Hom}(A_1(Y'), \mathbb{G}_m) \to \mathrm{Hom}(A_1(Y), \mathbb{G}_m) = T_Y$, an inclusion of tori. This identifies $T_{Y'}$ with the elements of T_Y which take the value 1 on exceptional divisors of τ. We define $\lambda' : \mathcal{Y}'_{\{E_{ij}\}} \to T_{Y'}$ to be the restriction of $\tilde{\mathcal{Y}}'_{\{E_{ij}\}}$ to $T_{Y'} \subset T_Y$. This inherits markings of the boundary and the Picard group. In this way we have a universal family associated with each configuration of exceptional curves for a toric model of some toric blowup.

Remark 5.8. Note in the construction we made a choice of right inverse $\gamma \colon T_{\bar{Y}} \to \mathrm{Aut}^0(D)$ of ψ. By Proposition 2.6, any two choices differ by a homomorphism $h \colon T_{\bar{Y}} \to \mathrm{Aut}(\bar{Y}, D)$. One can check that h together with the action of $\mathrm{Aut}(\bar{Y}, D)$ on $\bar{Y}$ induces a canonical identification of the universal families constructed.

Remark 5.9. There are in general infinitely many universal families of a given combinatorial type. For a given pair (Y, D) with exceptional divisors E_{ij} for a toric model, the above construction gives a finite number of families, as there is a choice of order of blowup. However, there may be an infinite number of sets of exceptional divisors of the same combinatorial type, giving rise to an infinite number of families. We will see that any two are birational, canonically identified by a birational map, see Construction-Theorem 5.12.

By construction:

Lemma 5.10. *For $\phi \in T_Y$, the marked period point of the fibre $((\mathcal{Y}, \mathcal{D}), p_i, \mu)_\phi$ of a universal family is ϕ.*

In particular, the fiber of a universal family $(\mathcal{Y},\mathcal{D})/T_{Y_0}$ over the identity $e \in T_{Y_0}$ is the pair (Y_e, D) defined above.

Corollary 5.11. *The locus of generic pairs in a universal family* $(\mathcal{Y},\mathcal{D})/T_Y$ *is the complement in* T_Y *of the countable union of hypertori*

$$T_\alpha = \{\phi \in T_Y \mid \phi(\alpha) = 1\}$$

for $\alpha \in \Phi$*.*

Proof. This follows from Corollary 3.5. □

We construct a birational action of Adm_{Y_0} on a universal family. This action is used to identify Adm_{Y_0} with the monodromy group, see Theorem 5.15.

Theorem-Construction 5.12. *Let* (Y_0, D) *be a generic Looijenga pair. Let* $\{E_{ij}\}, \{F_{ij}\}$ *be two exceptional configurations for* (Y_0, D)*, not necessarily of the same type. Then there is a canonical birational map*

$$\mathcal{Y}_{\{E_{ij}\}} \dashrightarrow \mathcal{Y}_{\{F_{ij}\}},$$

commuting with the projections to T_{Y_0}*. This birational map restricts to an isomorphism over a Zariski open set of* T_{Y_0} *containing the locus of generic pairs, and respects the markings of the Picard group and the boundary of each fiber over this locus.*

Proof. Suppose first the configurations are on Y_0 (rather than on possibly different toric blowups of Y_0). Let $U \subset T_{Y_0}$ be the maximal open subset such that each $F_{ij} \subset Y_0$ deforms to a family of (-1)-curves $\mathcal{F}_{ij} \subset \mathcal{Y}_{\{E_{ij}\}}|_U$ over U. The Zariski open set U contains the locus of generic fibers by Lemma 4.6. Write $\mathcal{Y}' = \mathcal{Y}_{\{E_{ij}\}}|_U$. The $\mathcal{F}_{ij}$ restrict to an exceptional configuration on each fiber of $\mathcal{Y}'$ over U, and we have a birational morphism $(\mathcal{Y}', \mathcal{D}') \to (\overline{\mathcal{Y}}', \overline{\mathcal{D}}')$ given by blowing down these families of curves. Let $\pi\colon (Y_0, D) \to (\bar{Y}_0, \bar{D})$ be the toric model obtained by contracting the F_{ij}. Then $(\overline{\mathcal{Y}}', \overline{\mathcal{D}}')$ is a fiber bundle over U with fiber $(\bar{Y}_0, \bar{D})$. Also, $\mathcal{Y}' \to \overline{\mathcal{Y}}'$ restricts to an isomorphism $\mathcal{D}' \to \overline{\mathcal{D}}'$, so the markings $p_i^E\colon T_{Y_0} \to \mathcal{Y}_{\{E_{ij}\}}$ induce markings of $\overline{\mathcal{D}}' \subset \overline{\mathcal{Y}}'$. Let p_i^F be the sections of the trivial family $U \times (\bar{Y}_0, \bar{D})$ given by Construction 5.7 for $\mathcal{Y}_{\{F_{ij}\}}$. By construction, the marked period points of the fibers of the families $((\overline{\mathcal{Y}}', \overline{\mathcal{D}}'), p_i^E)$ and $(U \times (\bar{Y}_0, \bar{D}), p_i^F)$ over each $\phi \in U$ coincide. So, by Lemma 2.7, there is a unique isomorphism $f : ((\overline{\mathcal{Y}}', \overline{\mathcal{D}}'), p_i^E) \to (U \times (\bar{Y}_0, \bar{D}), p_i^F)$ over U. Each of $\overline{\mathcal{Y}}'$ and $U \times \bar{Y}$ comes with sections q_{ij} (given in the first case by the images of the exceptional divisors $\mathcal{F}_{ij}$, in the second by Construction 5.7 for $\mathcal{Y}_{\{F_{ij}\}}$), which are identified under the isomorphism f. Thus after performing the iterated blow-up of the q_{ij} on $\overline{\mathcal{Y}}'$ and $U \times \bar{Y}_0$ respectively, f induces an isomorphism $\mathcal{Y}' \to \mathcal{Y}_{\{F_{ij}\}}|_U$. That is, we obtain a birational map $\mathcal{Y}_{\{E_{ij}\}} \dashrightarrow \mathcal{Y}_{\{F_{ij}\}}$ which is an isomorphism over U.

If the configurations are on toric blowups of Y_0 we make the obvious modifications: we replace Y_0 by a toric blowup $\tau : Z_0 \to Y_0$ on which they both appear and carry out the above. Then we restrict the birational maps to the subtorus $\mathrm{Hom}(\mathrm{Pic}(Y_0), \mathbb{G}_m) \subset \mathrm{Hom}(\mathrm{Pic}(Z_0), \mathbb{G}_m)$, and obtain induced birational maps between the universal families (which we recall are obtained from the restricted families via the families of toric blowdowns determined by τ). $\square$

Construction 5.13. Let (Y_0, D) be a generic Looijenga pair. Observe that $\mathrm{Aut}(\mathrm{Pic}(Y_0))$ acts by precomposition on T_{Y_0}:

$$g(\phi) := \phi \circ g^{-1}.$$

If g is admissible and $\{E_{ij}\}$ is an exceptional collection, then $\{g(E_{ij})\}$ is an exceptional collection necessarily of the same combinatorial type as $\{E_{ij}\}$. This induces, by the construction of the universal families, a commutative diagram

$$\begin{tikzcd} \mathcal{Y}_{\{E_{ij}\}} \arrow[r] \arrow[d] & \mathcal{Y}_{\{g(E_{ij})\}} \arrow[d] \\ T_{Y_0} \arrow[r, "\phi\mapsto\phi\circ g^{-1}"] & T_{Y_0} \end{tikzcd}$$

where the horizontal maps are isomorphisms. Composing $\mathcal{Y}_{\{E_{ij}\}} \to \mathcal{Y}_{\{g(E_{ij})\}}$ with the canonical birational map $\mathcal{Y}_{\{g(E_{ij})\}} \dashrightarrow \mathcal{Y}_{\{E_{ij}\}}$ gives a birational map

$$\psi_g : \mathcal{Y}_{\{E_{ij}\}} \dashrightarrow \mathcal{Y}_{\{E_{ij}\}}$$

which is an isomorphism over a Zariski open set containing the locus of generic pairs. In particular this gives a canonical action of Adm_{Y_0} on $\mathcal{Y}_{\{E_{ij}\}}$ by birational automorphisms. By construction the composition

$$\mathrm{Pic}(Y_0) \xrightarrow{r^{-1}} \mathrm{Pic}(\mathcal{Y}_{\{E_{ij}\}}) \xrightarrow{\psi_{g*}} \mathrm{Pic}(\mathcal{Y}_{\{E_{ij}\}}) \xrightarrow{r} \mathrm{Pic}(Y_0)$$

is $g \in \mathrm{Aut}(\mathrm{Pic}(Y_0))$ (here r is the restriction).

Example 5.14. Consider the pair (Y, D) obtained by blowing up one general point on each coordinate axis of $\mathbb{P}^2$, with D the proper transform of the toric boundary of $\mathbb{P}^2$. Write the generators of $\mathrm{Pic}(Y)$ as L, E_1, E_2, E_3 with L the pull-back of a line in $\mathbb{P}^2$ and the E_i's the exceptional divisors. Then $\{E_1, E_2, E_3\}$ is an exceptional configuration, as is $\{F_1, F_2, F_3\}$ where $F_i = (L - E_1 - E_2 - E_3) + E_i$. We obtain universal families $\mathcal{Y}_{\{E_{ij}\}}, \mathcal{Y}_{\{F_{ij}\}} \to T_Y$. The birational map constructed above $f : \mathcal{Y}_{\{E_{ij}\}} \dashrightarrow \mathcal{Y}_{\{F_{ij}\}}$ is an isomorphism away from the locus where the three blown-up points lie on a line $\overline{L}$. Over such a point, the curve of class F_i decomposes as a union of irreducible curves of class $\alpha := L - E_1 - E_2 - E_3$ and E_i. The curve of class α is the proper transform of $\overline{L}$ and is common to all three curves, hence the three curves cannot be simultaneously contracted. The proper transform of $\overline{L}$ must be flopped before this contraction can be performed.

Note in this example that $D^{\perp}$ is generated by α, and $\Phi = \{\pm\alpha\}$. The reflection s_α satisfies $s_\alpha(E_i) = F_i$. It is an admissible automorphism, and $W = \{\mathrm{id}, s_\alpha\}$. Since the only non-trivial automorphism which preserves the boundary classes and the intersection pairing is s_α, it is clear that $W = \mathrm{Adm}_Y$.

Using the construction of universal families together with the Adm_Y action, we show that Adm_Y is equal to the monodromy group.

Theorem 5.15. *Let (Y, D) be a Looijenga pair. The group Adm_Y is the monodromy group in the following sense. Let $(\mathcal{Y}, \mathcal{D}) \to S$ be a family of Looijenga pairs over a connected base S together with a point $s \in S$ and an identification*

$$(Y, D) \xrightarrow{\sim} (\mathcal{Y}_s, \mathcal{D}_s).$$

Then the monodromy map

$$\rho\colon \pi_1(S, s) \to \mathrm{Aut}(\mathrm{Pic}(Y))$$

has image contained in Adm_Y. Furthermore, in the analytic category, there exists a family as above such that S is smooth and the image of ρ is equal to Adm_Y.

Proof. We have already established that the monodromy group of any family is contained in Adm_Y. See Lemma 4.3. It remains to show that there is a family with smooth base and monodromy group equal to Adm_Y.

Let $\lambda\colon (\mathcal{Y}, \mathcal{D}) \to S$ be a choice of universal family, with marking

$$\mu\colon \mathrm{Pic}(Y) \times S \to R^2\lambda_*\mathbb{Z}$$

and action ψ of Adm_Y. Here $S = \mathrm{Hom}(\mathrm{Pic}(Y), \mathbb{G}_m)$. Let $A \subset \mathrm{Pic}(Y)_{\mathbb{R}}$ be a connected open cone on which Adm_Y acts properly discontinuously. For example we can take $A = C^+$. Working in the analytic topology, let $\Omega \subset S$ denote the *tube domain* associated to A. That is

$$\Omega = (\mathrm{Pic}(Y)_{\mathbb{R}} + iA)/\operatorname{Pic}(Y) \subset (\mathrm{Pic}(Y) \otimes_{\mathbb{Z}} \mathbb{C})/\operatorname{Pic}(Y) = S,$$

an open analytic subset of S. (Here we have used the identification $\mathrm{Pic}(Y) = \mathrm{Pic}(Y)^*$ given by Poincaré duality.) Then Adm_Y acts properly discontinuously on Ω. (Indeed, the real analytic morphism

$$\Omega \to A$$

given by projection onto the imaginary part is Adm_Y equivariant and Adm_Y acts properly discontinuously on A.) Let

$$\Omega^o := \Omega \setminus \bigcup_{g \in \mathrm{Adm}_Y} \mathrm{Fix}(g)$$

denote the complement of the fixed loci of the elements of Adm_Y. Note that

$$\bigcup_{g \in \mathrm{Adm}_Y} \mathrm{Fix}(g) \subset \Omega$$

is a locally finite union of analytic subvarieties because the action is properly discontinuous. Hence $\Omega^o \subset \Omega$ is a connected open analytic subset. Note also that Ω^o is contained in the locus of generic pairs. Indeed, the locus of generic pairs is the complement of union of the hypertori T_α for $\alpha \in \Phi$ by Corollary 5.11, and $T_\alpha = \mathrm{Fix}(s_\alpha)$ where $s_\alpha \in W \subset \mathrm{Adm}_Y$ is the reflection in the root α. Let

$$U := \Omega^o / \mathrm{Adm}_Y$$

be the quotient of Ω^o by Adm_Y, a complex analytic manifold. Let $(\mathcal{Y}_U, \mathcal{D}_U) \to U$ be the family of Looijenga pairs over U given by the quotient of the restriction of the family $(\mathcal{Y}, \mathcal{D}) \to S$ to Ω^o. (Note that the birational action of Adm_Y on the universal family is biregular over the locus of generic pairs and hence over Ω^o. See Construction 5.13.) Let $t \in \Omega^o$ be a basepoint, and $u \in U$ the image of t. The Galois covering map $\Omega^o \to U$ with group Adm_Y corresponds to a surjection

$$\pi_1(U, u) \to \mathrm{Adm}_Y .$$

Given $g \in \mathrm{Adm}_Y$, let $[\gamma] \in \pi_1(U, u)$ be a lift of g. Then γ is a loop based at $u \in U$ which lifts to a path $\tilde{\gamma}$ in Ω^o from t to $g^{-1}t$. Now the monodromy transformation associated to the loop γ for the family $(\mathcal{Y}_U, \mathcal{D}_U) \to U$ is identified with $g \in \mathrm{Adm}_Y \subset \mathrm{Aut}(\mathrm{Pic}(Y))$ via the marking isomorphism

$$\mu_t \colon \mathrm{Pic}(Y) \xrightarrow{\sim} \mathrm{Pic}(\mathcal{Y}_t).$$

$\square$

6. Moduli stacks

We give a complete description of the moduli stacks of Looijenga pairs, with and without markings. Note that these stacks are highly non-separated in general. The situation is very similar to that of moduli of K3 surfaces without polarization, cf. [LP80], §10.

We work in the analytic category. The stacks we define are stacks over the category of analytic spaces.

Fix a Looijenga pair (Y_0, D). Let $\tilde{\mathbb{M}}_{Y_0}$ denote the moduli stack of families of Looijenga pairs (Y, D) together with a marking of D and a marking of $\mathrm{Pic}(Y)$ by $\mathrm{Pic}(Y_0)$. More precisely, for an analytic space S, the objects of the category $\tilde{\mathbb{M}}_{Y_0}(S)$ are morphisms

$$\lambda\colon (\mathcal{Y}, \mathcal{D} = \mathcal{D}_1 + \cdots + \mathcal{D}_n) \to S$$

together with an isomorphism

$$\mu\colon \mathrm{Pic}(Y_0) \times S \xrightarrow{\sim} R^2\lambda_*\mathbb{Z}$$

and sections

$$p_i\colon S \to \mathcal{D}_i$$

of $\mathcal{D}_i \to S$ such that

(1) $\mathcal{Y}/S$ is a flat family of surfaces,
(2) $\mathcal{D}_i$ is a Cartier divisor on $\mathcal{Y}/S$ for each i, and
(3) Each closed fiber $(\mathcal{Y}_s, \mathcal{D}_s, \mu_s, \{p_i(s)\})$ is a Looijenga pair together with marking $\mu_s\colon \mathrm{Pic}(Y_0) \to \mathrm{Pic}(\mathcal{Y}_s)$ of the Picard group and marking $p_i(s) \in \mathcal{D}_{i,s}$ of the boundary.

Furthermore, in the cases $n = 1$ or 2, we assume given an orientation of $\mathcal{D}$, that is, an identification

$$\mathbb{Z} \times S \xrightarrow{\sim} R^1\lambda_*\mathbb{Z}_{\mathcal{D}}.$$

The morphisms in the category from $(\mathcal{Y}, \mathcal{D})/S$ to $(\mathcal{Y}', \mathcal{D}')/S'$ over a morphism $S \to S'$ are isomorphisms

$$(\mathcal{Y}, \mathcal{D}) \xrightarrow{\sim} (\mathcal{Y}', \mathcal{D}') \times_{S'} S$$

over S compatible with the markings and the orientation.

Similarly, let $\tilde{\mathbb{M}}'_{Y_0}$ denote the moduli stack of Looijenga pairs (Y, D) together with a marking of $\mathrm{Pic}(Y)$ by $\mathrm{Pic}(Y_0)$, $\mathbb{M}_{Y_0}$ the moduli stack of Looijenga pairs with a marking of D, and $\mathbb{M}'_{Y_0}$ the moduli stack of Looijenga pairs.

We have the period mapping

$$\tilde{\mathbb{M}}_{Y_0} \to T_{Y_0}$$

$$((Y, D), \mu, \{p_i\}) \mapsto \phi_Y.$$

(Note: If $(\mathcal{Y}, \mathcal{D})/S$ is an object of $\tilde{\mathbb{M}}_{Y_0}$ then the sections p_i and the orientation of $\mathcal{D}$ determine a canonical isomorphism

$$D \times S \xrightarrow{\sim} \mathcal{D}.$$

This is used to define the period mapping for families over an arbitrary base S.) Similarly, writing $T'_{Y_0} = \mathrm{Hom}(D^\perp, \mathbb{G}_m)$, we have period mappings

$$\tilde{\mathbb{M}}'_{Y_0} \to T'_{Y_0},$$

$$\mathbb{M}_{Y_0} \to [T_{Y_0}/\operatorname{Adm}],$$

and

$$\mathbb{M}'_{Y_0} \to [T'_{Y_0}/\operatorname{Adm}].$$

(Here for a group G acting on an analytic space X we write $[X/G]$ for the stack quotient. We also write $\operatorname{Adm} = \operatorname{Adm}_{Y_0}$ for brevity.)

Let Σ denote the set of connected components of the complement

$$C_D^{++} \setminus \bigcup_{\alpha\in\Phi} \alpha^{\perp}.$$

(So Σ is permuted simply transitively by the Weyl group W.) Let $U = T_{Y_0} \times \Sigma$. Define an étale equivalence relation R on U as follows: $(p,\sigma) \sim (p,\sigma')$ iff σ and σ' are contained in the same connected component of $C_D^{++} \setminus \bigcup_{\alpha\in\Phi_p} \alpha^{\perp}$, where

$$\Phi_p := \{\alpha \in \Phi \mid p(\alpha) = 1\}.$$

Let $\tilde{T}_{Y_0}$ denote the analytic space U/R. Thus we have an étale morphism $\tilde{T}_{Y_0} \to T_{Y_0}$ given by the first projection $U = T_{Y_0} \times \Sigma \to T_{Y_0}$, which is an isomorphism over the open analytic set $T_{Y_0} \setminus \overline{\bigcup_{\alpha\in\Phi}(p(\alpha)=1)}$. Note that $\tilde{T}_{Y_0}$ is not separated if $\Phi \neq \emptyset$. We define $\tilde{T}'_{Y_0} \to T'_{Y_0}$ similarly. That is, $\tilde{T}'_{Y_0} = U'/R'$ where $U' = T'_{Y_0} \times \Sigma$ and R' is defined by the same rule as above.

Recall the action of $\operatorname{Aut}^0(D) = \mathbb{G}_m^n$ on

$$T_{Y_0} = \operatorname{Hom}(\operatorname{Pic}(Y_0), \mathbb{G}_m) = \operatorname{Hom}(\operatorname{Pic}(Y_0), \operatorname{Pic}^0(D))$$

from Lemma 2.5.

Let $K = \operatorname{Hom}(N', \mathbb{G}_m)$ where N' is the group defined in Theorem 1.8. Every object $(\mathcal{Y}, \mathcal{D})/S$ of $\tilde{\mathbb{M}}'_{Y_0}(S)$ has a canonical subgroup $K \times S$ of its automorphism group. (These are the automorphisms acting trivially on the Picard group of the fibers). See Theorem 5.1. Let $\tilde{\mathbb{M}}''_{Y_0}$ denote the rigidification of $\tilde{\mathbb{M}}'_{Y_0}$ along K in the sense of [ACV05], §5. (Thus the objects of $\tilde{\mathbb{M}}''_{Y_0}$ and $\tilde{\mathbb{M}}'_{Y_0}$ coincide locally, but the automorphism group in $\tilde{\mathbb{M}}''_{Y_0}$ is the quotient of the automorphism group in $\tilde{\mathbb{M}}'_{Y_0}$ by K.) Similarly let $\mathbb{M}''_{Y_0}$ denote the rigidification of $\mathbb{M}'_{Y_0}$ along K. (Note that the monodromy group Adm acts trivially on K.) The period mappings $\tilde{\mathbb{M}}'_{Y_0} \to T'_{Y_0}$ and $\mathbb{M}'_{Y_0} \to [T'_{Y_0}/\operatorname{Adm}]$ descend to maps $\tilde{\mathbb{M}}''_{Y_0} \to T'_{Y_0}$ and $\mathbb{M}''_{Y_0} \to [T'_{Y_0}/\operatorname{Adm}]$.

Theorem 6.1. *We have identifications*

$$\tilde{\mathbb{M}}_{Y_0} = \tilde{T}_{Y_0},$$

$$\tilde{\mathbb{M}}'_{Y_0} = [\tilde{T}_{Y_0}/\operatorname{Aut}^0(D)],$$

$$\mathbb{M}_{Y_0} = [\tilde{T}_{Y_0}/\operatorname{Adm}],$$

$$\mathbb{M}'_{Y_0} = [\tilde{T}_{Y_0}/\operatorname{Aut}^0(D) \times \operatorname{Adm}],$$

$$\tilde{\mathbb{M}}''_{Y_0} = \tilde{T}'_{Y_0},$$

and

$$\mathbb{M}''_{Y_0} = [\tilde{T}'_{Y_0} / \operatorname{Adm}],$$

compatible with the period mappings.

Proof. The identification

$$\tilde{\mathbb{M}}_{Y_0} \xrightarrow{\sim} \tilde{T}_{Y_0}$$

is obtained as follows. Given $((\mathcal{Y}, \mathcal{D})/S, \mu, \{p_i\}) \in \tilde{\mathbb{M}}_{Y_0}(S)$ we have the associated period mapping $\phi\colon S \to T_{Y_0}$, see Definition 1.11(3). We define a lift $\tilde{\phi}\colon S \to \tilde{T}_{Y_0}$ of ϕ by

$$\tilde{\phi}(s) = (\phi(s), \sigma)$$

where $\sigma \subset \mu_s^{-1}(\operatorname{Nef}(\mathcal{Y}_s))$, for each $s \in S$. Note that $\mu_s^{-1}(\operatorname{Nef}(\mathcal{Y}_s))$ is the closure of a connected component of $C_D^{++} \setminus \bigcup_{\alpha \in \Phi_p} \alpha^\perp$ by Lemma 2.15 and Theorem 3.2. So σ is uniquely determined up to the equivalence relation R and $\tilde{\phi}$ is a well-defined map to $\tilde{T}_{Y_0} = U/R$.

We now establish that the morphism $\tilde{\mathbb{M}}_{Y_0} \to \tilde{T}_{Y_0}$ is an isomorphism. Objects of $\tilde{\mathbb{M}}_{Y_0}$ have no non-trivial automorphisms by Proposition 2.6. So the stack $\tilde{\mathbb{M}}_{Y_0}$ is (represented by) an analytic space. The period mapping $\tilde{\mathbb{M}}_{Y_0} \to T_{Y_0}$ is étale by [L81], II.2.5. Hence also $\tilde{\mathbb{M}}_{Y_0} \to \tilde{T}_{Y_0}$ is étale. The map $\tilde{\mathbb{M}}_{Y_0} \to \tilde{T}_{Y_0}$ is injective on points by the global Torelli theorem for Looijenga pairs, Theorem 1.8. Indeed, two marked Looijenga pairs $((Y, D), \mu, \{p_i\})$, $((Y', D'), \mu', \{p'_i\})$ are isomorphic iff $\phi_Y = \phi_{Y'}$ and $\mu^{-1}(\operatorname{Nef}(Y)) = \mu'^{-1}(\operatorname{Nef}(Y'))$ by Remark 1.9. Also, the map is surjective on points by the construction of universal families (see Lemma 5.10) and the fact that connected components of $C_D^{++} \setminus \bigcup_{\alpha \in \Phi_p} \alpha^\perp$ are permuted transitively by the Weyl group $W(\Phi_p)$. Hence the map $\tilde{\mathbb{M}}_{Y_0} \to \tilde{T}_{Y_0}$ is an isomorphism as claimed.

The remaining identifications follow by passing to the quotients corresponding to forgetting the marking of $\operatorname{Pic}(Y)$ and/or D. Note that applying $\operatorname{Hom}(\cdot, \mathbb{G}_m)$ to the exact sequence

$$0 \to D^\perp \to \operatorname{Pic}(Y_0) \to \mathbb{Z}^n \to N' \to 0$$

we obtain the exact sequence

$$1 \to K \to \mathbb{G}_m^n \to T_{Y_0} \to T'_{Y_0} \to 1.$$

Hence

$$\tilde{\mathbb{M}}'_{Y_0} = [\tilde{T}_{Y_0} / \operatorname{Aut}^0(D)] = [\tilde{T}_{Y_0} / \mathbb{G}_m^n],$$

where $K \subset \mathbb{G}_m^n$ acts trivially and the quotient $H := \mathbb{G}_m^n / K$ acts freely. Thus rigidifying $\tilde{\mathbb{M}}'_{Y_0}$ along K yields $[\tilde{T}_{Y_0}/H] = \tilde{T}'_{Y_0}$. □

7. Generalization of the Tits cone

In this section we explore to what extent some additional constructions from [L81] extend to the more general context of this paper.

The paper [L81] considers Looijenga pairs (Y, D) such that the following conditions are satisfied:

Assumptions 7.1. (1) *The number n of irreducible components of D is less than or equal to 5.*
(2) *The intersection matrix $(D_i \cdot D_j)$ is negative semi-definite.*
(3) *There do not exist (-1)-curves contained in D.*

(Note that condition (3) is not essential: Under condition (2), there is always a toric blowdown $(Y, D) \to (Y', D')$ such that (Y', D') satisfies (3).)

Remark 7.2. Under assumptions 7.1, Looijenga gives an explicit description of the set Δ and shows that it is a basis of the lattice $D^\perp := \langle D_1, \ldots, D_n\rangle^\perp$. In general however the set Δ does not give a basis of $D^\perp$. In fact Δ may be infinite, see Example 5.5. At the other extreme, there are examples with $D^\perp \neq 0$ and $\Delta = \emptyset$, see e.g. [F13], Examples 4.3 and 4.4.

Under assumptions 7.1, Looijenga defines the Tits cone $I \subset \operatorname{Pic}(Y)_{\mathbb{R}}$ as follows. (Here we use our notation.) Write $\Delta = \Delta_{Y_e}$ for the set of classes of (-2)-curves on Y_e. Define the fundamental chamber

$$C = \{x \in C^+ \mid x \cdot \alpha > 0 \text{ for all } \alpha \in \Delta\}.$$

The Tits cone I is defined by

$$I = \bigcup_{w \in W} w(\overline{C}).$$

Looijenga proves that the Weyl group W acts properly discontinuously on the interior $\operatorname{Int}(I)$ of I [L81], Corollary 1.14. Moreover, the reflection hyperplanes $\alpha^\perp \subset \operatorname{Pic}(Y)_{\mathbb{R}}$, $\alpha \in \Phi$ are dense in $\operatorname{Pic}(Y)_{\mathbb{R}} \setminus \operatorname{Int}(I) \cup (-\operatorname{Int}(I))$ by [L81], Theorem II.1.5. So

$$\operatorname{Int}(I) \cup (-\operatorname{Int}(I)) \subset \operatorname{Pic}(Y)_{\mathbb{R}}$$

is the maximal W-equivariant open set on which W acts properly discontinuously.

In the general case recall that we have an inclusion $W \subset \operatorname{Adm}$. The group Adm is the full monodromy group and the Weyl group W is the normal subgroup given by Picard–Lefschetz transformations. Under assumptions 7.1 we have $W = \operatorname{Adm}$, see Lemma 5.4. In general $W \neq \operatorname{Adm}$, and in fact the index of $W \subset \operatorname{Adm}$ may be infinite, see Example 5.3. Moreover, there are examples such that W is trivial and Adm is infinite, see Example 5.6.

However, we show that the fact that $W = \mathrm{Adm}$ acts properly discontinuously on the Tits cone admits a generalization as follows.

Proposition 7.3. *Assume the conditions 7.1. Let (Y_g, D) denote a generic deformation of (Y, D). Then the closure of the Tits cone $I \subset \mathrm{Pic}(Y)_{\mathbb{R}}$ is equal to the closure of*

$$\overline{\mathrm{NE}}(Y_g) + \langle D_1, \ldots, D_n \rangle_{\mathbb{R}}.$$

Proof. Write $A = \overline{\mathrm{NE}}(Y_g) + \langle D_1, \ldots, D_n \rangle_{\mathbb{R}}$, a convex cone in $\mathrm{Pic}(Y)_{\mathbb{R}}$. We must show that $\overline{A} = \overline{I}$.

The Weyl group W acts transitively on the set of (-1)-curves on Y_g meeting D_i, for each $i = 1, \ldots, n$, by [L81] Theorem I.4.6. Also $C^+ \subset I$ by [L81], Lemma I.3.7.

If $D^2 < 0$ then I is the convex hull of the union of $\langle D_1, \ldots, D_n \rangle_{\mathbb{R}}$ and the set of (-1)-curves on Y_g by [L81] Proposition I.3.9, the description of the extremal facets of I in §I.3.8, and Theorem I.4.6. Now by the description of $\overline{\mathrm{NE}}(Y_g)$ given by Lemma 2.14 we deduce that $\overline{I} = \overline{A}$ if $D^2 < 0$.

If $D^2 = 0$ then

$$I = \{x \in \mathrm{Pic}(Y)_{\mathbb{R}} \mid x \cdot D > 0\} \cup \langle D_1, \ldots, D_n \rangle_{\mathbb{R}}$$

by [L81] Proposition I.3.9. It is easy to see that $\overline{I} = \overline{A}$ in this case. Indeed, if $x \cdot D > 0$ then $(x + ND) \in C^+$ for $N \gg 0$, thus $x \in A$. So $I \subset A$. Conversely, $A \subset \overline{I}$ because D is effective and $D \cdot D_i = 0$ for each component D_i of D (note that D is either irreducible with $D^2 = 0$ or a cycle of (-2)-curves). □

Proposition 7.4. *Let Y be a smooth projective surface. Let $\Gamma \subset \mathrm{Aut}(\mathrm{Pic}(Y))$ be a subgroup preserving the semigroup of effective classes. Then Γ acts properly discontinuously on the interior of the cone $\overline{\mathrm{NE}}(Y)$.*

Proof. First note that any subgroup Γ of $\mathrm{Aut}(\mathrm{Pic}(Y))$ acts properly discontinuously on the positive cone C^+.

Now assume as in the statement that Γ preserves the semigroup of effective classes. We will use the Zariski decomposition of effective divisors on the surface Y to show that Γ acts properly discontinuously on the interior of the effective cone. Let D be a pseudoeffective $\mathbb{R}$-divisor on the surface Y (that is, $D \in \overline{\mathrm{NE}}(Y)$). Then there is a unique decomposition

$$D = P + N$$

where P and N are $\mathbb{R}$-divisors, P is nef, N is effective, and, writing $N = \sum a_i N_i$ where N_i is irreducible and $a_i \in \mathbb{R}_{>0}$ for each i, we have $P \cdot N_i = 0$ for each i and the matrix $(N_i \cdot N_j)$ is negative definite. See [KMM87], Theorem 7.3.1. Moreover D lies in the interior of the effective cone iff $P^2 > 0$. (Indeed $C^+ \subset \overline{\mathrm{NE}}(Y)$ so $P^2 > 0$ implies D lies in the interior of $\overline{\mathrm{NE}}(Y)$. Conversely if $P^2 = 0$ and $P \neq 0$ then $D \in P^{\perp}$ and P is nef so D does not lie in the interior of $\overline{\mathrm{NE}}(Y)$. Finally if $P = 0$ then we can find a nef divisor B such that $B \cdot D = 0$ — take an ample divisor A and write $B = A + \sum \lambda_i N_i$ such that $B \cdot N_i = 0$ for each i, then $\lambda_i > 0$ for each i and B is nef, $B \cdot D = 0$. Thus D does not lie in the interior of $\overline{\mathrm{NE}}(Y)$.) Note also that if D lies in the interior of the effective cone then the Zariski decomposition $D = P + N$ is characterized by the following properties: P is nef, $P \neq 0$, N is effective, and $P \cdot N = 0$. (Indeed, writing $N = \sum_{i=1}^{k} a_i N_i$ as above, $P \cdot N = 0$ and P nef implies $P \cdot N_i = 0$ for each i. Also $P^2 > 0$ because D lies in the interior of the effective cone, so the subspace $\langle N_1, \ldots, N_k \rangle_{\mathbb{R}} \subset P^{\perp}$ is negative definite. It remains to show that the N_i are linearly independent. Otherwise, we have a nontrivial expression $\sum \alpha_i N_i = \sum \beta_i N_i$ where $\alpha_i, \beta_i \in \mathbb{R}_{\geq 0}$ and $\alpha_i \beta_i = 0$ for each i. But then

$$0 > (\sum \alpha_i N_i)^2 = (\sum \alpha_i N_i) \cdot (\sum \beta_i N_i) \geq 0,$$

a contradiction.)

Let $B \subset \mathrm{Pic}(Y)_{\mathbb{R}}$ denote the interior of the effective cone. We need to show that Γ acts properly discontinuously on B. Equivalently, the map

$$\Gamma \times B \to B \times B, \quad (\gamma, x) \mapsto (x, \gamma x)$$

is proper (that is, the inverse image of a compact set is compact). Equivalently, if (γ_n, x_n) is a sequence in $\Gamma \times B$ such that $x_n \to x$ and $\gamma_n x_n \to y$ as $n \to \infty$ for some $x, y \in B$, then $\gamma_n = \gamma$, for some $\gamma \in \Gamma$, for infinitely many n. Let $x_n = P_n + N_n$, $x = P + N$, and $y = P' + N'$ be the Zariski decompositions of x_n, x, and y. Then $P_n \to P$ and $N_n \to N$ as $n \to \infty$ by continuity of the Zariski decomposition on the interior of the effective cone [BKS04], Proposition 1.16. Also, since by assumption Γ preserves the semigroup of effective classes, $\gamma_n x_n = \gamma_n P_n + \gamma_n N_n$ is the Zariski decomposition of $\gamma_n x_n$. Thus $\gamma_n P_n \to P'$ and $\gamma_n N_n \to N'$ as $n \to \infty$. Now $P_n \to P$ and $\gamma_n P_n \to P'$ implies $\gamma_n = \gamma$, some $\gamma \in \Gamma$, for infinitely many n because Γ acts properly discontinuously on C^+. ☐

Lemma 7.5. *Let (Y, D) be a Looijenga pair. Let (Y_g, D) be a generic deformation of (Y, D). Then the monodromy group Adm_Y preserves the semigroup of effective classes on (Y_g, D).*

Proof. By Lemma 2.14(1), if $C \subset Y_g$ is an irreducible curve, then either $C \subset D$, $C^2 \geq 0$, or C is a (-1)-curve. The group Adm_Y preserves the boundary classes $[D_i]$ and the ample cone of Y_g by Lemma 2.15. It follows from Riemann–Roch that $\mu(C)$ is effective for $\mu \in \mathrm{Adm}_Y$ and C either an irreducible curve such that $C^2 \geq 0$ or a (-1)-curve. So Adm_Y preserves the semigroup of effective classes. ☐

Corollary 7.6. *Let (Y, D) be a Looijenga pair. Let (Y_g, D) be a generic deformation of (Y, D). Then* Adm_Y *acts properly discontinuously on the interior of* $\overline{\mathrm{NE}}(Y_g) + \langle D_1, \ldots, D_n\rangle_{\mathbb{R}}$.

Proof. The group Adm_Y acts properly discontinuously on the interior of $\overline{\mathrm{NE}}(Y_g)$ by Lemma 7.5 and Proposition 7.4. Since Adm_Y acts trivially on the subspace $\langle D_1, \ldots, D_n\rangle_{\mathbb{R}}$, it follows that Adm_Y acts properly discontinuously on the interior of $\overline{\mathrm{NE}}(Y_g)+\langle D_1, \ldots, D_n\rangle_{\mathbb{R}}$. (Indeed any two points x, y in the interior of $\overline{\mathrm{NE}}(Y_g) + \langle D_1, \ldots, D_n\rangle_{\mathbb{R}}$ are contained in a translate T of the interior of $\overline{\mathrm{NE}}(Y_g)$ by some element $z = \sum a_i D_i$, $a_i \in \mathbb{R}$. Thus there exist open neighbourhoods $x \in U \subset T$ and $y \in V \subset T$ such that the set $\{g \in \mathrm{Adm}_Y \mid gU \cap V \neq \emptyset\}$ is finite because Adm_Y acts properly discontinuously on T.) □

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