508A Lecture notes

Paul Hacking

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This document contains lecture notes for 508A, a second course in Algebraic geometry at the University of Washington, in Winter quarter 2008.

We always work over an algebraically closed field k. We often assume $k = \mathbb{C}$.

1 What is an algebraic variety?

First, compare definition of a smooth manifold:

X connected topological space.

Charts: $X = \bigcup_{i \in I} U_i$ countable open covering. $\phi_i \colon U_i \to V_i \subset \mathbb{R}^n$ homeomorphism of U_i onto open subset $V_i \subset \mathbb{R}^n$. Here n is the *dimension* of X.

We require that the charts are compatible in the following sense: the transition function $\psi_{ij} := \phi_j \circ \phi_i^{-1}$, a homeomorphism between the open subsets $\phi_i(U_i \cap U_j) \subset V_i$ and $\phi_j(U_i \cap U_j) \subset V_j$, is a diffeomorphism for all i, j.

We also require that the topological space X is *Hausdorff*, that is, for all $x, y \in X$, $x \neq y$, there exist open neighbourhoods $x \in U$, $y \in V$ such that $U \cap V = \emptyset$.

Now, define algebraic variety:

X connected topological space (think: Zariski topology).

Charts: $X = \bigcup U_i$ *finite* open covering. $\phi_i \colon U_i \to V_i \subset \mathbb{A}^{n_i}$ homeomorphism of U_i onto closed irreducible subset $V_i \subset \mathbb{A}^{n_i}$. So V_i is an affine variety with its Zariski topology.

We require that the transition functions ψ_{ij} are *regular*, that is, regular functions on $\phi_j(U_i \cap U_j) \subset V_j$ pullback to regular functions on $\phi_i(U_i \cap U_j) \subset V_i$ via ψ_{ij}. (Note that we could have stated the compatibility condition for charts of smooth manifolds this way too: we require that smooth functions pullback to smooth functions).

We also require an analogue of the Hausdorff condition. We cannot use the usual definiton of Hausdorff because the Zariski topology is too coarse (has too few open sets). Instead we require that X is *separated*: that is, for Z an affine variety and $f\colon Z \to X$, $g\colon Z \to X$ two morphisms, the locus $(f = g) \subset Z$ is closed.

Example 1.1. Let X be obtained by glueing two copies of $\mathbb{A}^1$ along the open subset $\mathbb{A}^1 - \{0\}$. Then X is not separated.

Example 1.2. If $k = \mathbb{C}$, we can associate to X an *analytic space* X^{an}. The topology on X^{an} is induced by the usual Euclidean topology on each $V_i \subset \mathbb{C}^{n_i}$. (We can also define the analytic functions on X as those which are holomorphic in the charts $V_i \subset \mathbb{C}^{n_i}$, equivalently, can be locally expanded in power series in the coordinates on $\mathbb{C}^{n_i}$). Then X is separated iff X^{an} is Hausdorff. This follows from [Mumford, p. 114, Cor. 1].

A *morphism* of varieties $f\colon X \to Y$ is a continuous map of topological spaces which is given by a morphism of open subsets of affine varieties in charts.

We can specify an algebraic variety as follows: we have a topological space X, and a 'sheaf' $\mathcal{O}_X$ of regular functions on X (the *structure sheaf*). That is, for every open $U \subset X$, we have a set $\mathcal{O}_X(U)$ of regular functions on U, which is a subset of the set of all continuous functions $U \to \mathbb{A}^1_k$ (where we use the Zariski topology). In charts $\phi\colon U_i \to V_i \subset \mathbb{A}^{n_i}$ as above, a regular function on an open subset $W \subset X$ is a function whose restriction to $U_i \cap W$ corresponds to a regular function on $\phi_i(U_i \cap W) \subset V_i$ for each i. From this point of view, a morphism $f\colon X \to Y$ is a continuous map of topological spaces such that regular functions pullback to regular functions.

Example 1.3. Projective space $\mathbb{P}^N$ is covered by $N + 1$ open affine subsets $U_i = (X_i \neq 0) = \mathbb{A}^N$. Here $X_0, .., X_N$ are the homogeneous coordinates on projective space, and the affine coordinates on $U_i = \mathbb{A}^N$ are given by X_j/X_i for $j \neq i$. If now $X \subset \mathbb{P}^N$ is a closed subvariety, we get a cover of X by the open affines $U_i \cap X$.

Although the abstract definition of a variety is important, most varieties of interest are projective (or quasiprojective — that is, an open subvariety of a projective variety).

2 Curves

A *curve* is a variety X over k of dimension 1.

(Recall $\dim X = 1$ is equivalent to any of the following: (1) the only closed subsets of X are X itself and finite subsets (2) the transcendence degree of $k(X)$ over k equals 1 (3) if $P \in X$ is a smooth point of X, then the local ring $\mathcal{O}_{X,P}$ of regular functions at P is a discrete valuation ring (4) if $P \in X$ is a smooth point then the completion $\hat{\mathcal{O}}_{X,P}$ is isomorphic to $k[[t]]$, the power series ring in one variable. Note that (4) is the closest to the usual definition for smooth manifolds).

We have a (contravariant) equivalence between the following categories:

ALGEBRA

Objects: K/k field extension which is finitely generated and of transcendence degree 1.

Morphisms: Inclusions $K \subset L$ of extensions of k.

GEOMETRY

Objects: X smooth projective curve over k.

Morphisms: $f \colon X \to Y$ nonconstant morphism.

The GEOMETRY to ALGEBRA functor is the following:

$X \mapsto K = k(X)$, the function field of X.

$(f \colon X \to Y) \mapsto f^\sharp \colon k(Y) \subset k(X)$, where $f^\sharp(g) = g \circ f$.

Constructing the inverse functor is more difficult. Here is a sketch, see [Hartshorne, I.6.12] for more details.

(Recall that a *discrete valuation* of K/k is a map $\nu \colon K^\times \to \mathbb{Z}$ such that $\nu(xy) = \nu(x) + \nu(y)$, $\nu(x+y) \geq \min(\nu(x), \nu(y))$, and $\nu(\lambda) = 0$ for $\lambda \in k^\times$. The associated *discrete valuation ring* (DVR) of K/k is

$$R = \{\nu \geq 0\} \cup \{0\} \subset K.$$

If X is a curve and $P \in X$ is a point then $\mathcal{O}_{X,P} \subset k(X)$ is a DVR of $k(X)/k$ with valuation $\nu_P \colon k(X)^\times \to \mathbb{Z}$ given by the order of vanishing at $P \in X$).

Given K, we construct X as follows.

As a set, $X = \{R \mid R \text{ a DVR of } K/k\}$.

As a topological space: The open subsets of X are the empty set and the complements of finite subsets.

As a variety: $\mathcal{O}_X(U) := \bigcap_{R \in U} R$. An element $f \in \mathcal{O}_X(U)$ defines a function $f \colon U \to k$ by $f(R) = \bar{f}$, where $\bar{f} \in k$ is the image of f in the residue field $R/m_R = k$ of R.

Given $K \subset L$, we construct $f \colon X \to Y$. As a map of sets, $R \mapsto R \cap K$. Then f is continuous, and regular functions pullback to regular functions, so f is a morphism.

Remark 2.1. In higher dimensions, it is *not* true that for smooth projective varieties X, Y, $k(X) \simeq k(Y)$ implies $X \simeq Y$. Indeed, $k(X) \simeq k(Y)$ iff X and Y are *birational*, that is, there exist open sets $U \subset X$ and $V \subset Y$ and an isomorphism $U \simeq V$. Now if X is a smooth projective variety of dimension ≥ 2 and $P \in X$ is a point, the blowup of $P \in X$ is a smooth projective variety $\tilde{X}$ which is birational to X but not isomorphic to X.

3 Morphisms of Curves

Proposition 3.1. *Let X be a projective variety, C a smooth curve, and $f\colon C \dashrightarrow X$ a rational map. Then f is a morphism.*

(Recall that a *rational map* $f\colon X \dashrightarrow Y$ is an equivalence class of morphisms $g\colon U \to Y$, where $U \subset X$ is a nonempty open set. $g\colon U \to Y$ and $h\colon V \to Y$ are equivalent if $g = h$ on $U \cap V$.)

Proof. Write $f = (f_0 : \cdots : f_N)\colon C \dashrightarrow X \subset \mathbb{P}^N$ where the f_i are rational functions on X. Let $P \in C$. Let $m_i = \nu_P(f_i)$, the order of vanishing of f_i at P. Let $m = \min m_i$ and $g_i = t^{-m} f_i$ where t is a local parameter at $P \in C$ (a generator of the maximal ideal $m_{C,P} \subset \mathcal{O}_{C,P}$). Then $f = (g_0 : \cdots : g_N)$ is well defined at P — because each g_i is regular at P and some g_i is nonzero at P. $\square$

Remark 3.2. This is false in higher dimensions: If X is a projective variety, Y is a smooth variety, and $f\colon Y \dashrightarrow X$ is a rational map, then f is not a morphism in general. For example consider the blowup $\pi\colon \mathrm{Bl}_0 \mathbb{A}^2_{x,y} \to \mathbb{A}^2_{x,y}$ of $0 \in \mathbb{A}^2_{x,y}$. The variety $\mathrm{Bl}_0 \mathbb{A}^2_{x,y}$ is a union of two affine pieces U_1, U_2 as follows.

$$U_1 = \mathbb{A}^2_{x',u} \to \mathbb{A}^2_{x,y},\ (x', u) \mapsto (ux', u)$$

$$U_2 = \mathbb{A}^2_{v,y'} \to \mathbb{A}^2_{x,y},\ (v, y') \mapsto (v, vy')$$

The glueing is given by

$$U_1 \supset (x' \neq 0) \xrightarrow{\sim} (y' \neq 0) \subset U_2,\ (x', u) \mapsto (v, y') = (ux', x'^{-1})$$

(Alternatively, $\mathrm{Bl}_0 \mathbb{A}^2_{x,y} = (xY - yX = 0) \subset \mathbb{A}^2_{x,y} \times \mathbb{P}^1_{(X:Y)} \xrightarrow{\mathrm{pr}_1} \mathbb{A}^2_{x,y}$).

The map $\pi\colon \mathrm{Bl}_0 \mathbb{A}^2_{x,y} \to \mathbb{A}^2_{x,y}$ satisfies $\pi^{-1}(0) = E \simeq \mathbb{P}^1$, the *exceptional curve*, and π restricts to an isomorphism $\mathrm{Bl}_0 \mathbb{A}^2_{x,y} \setminus E \xrightarrow{\sim} \mathbb{A}^2_{x,y} \setminus \{0\}$. E is given in the charts by $(u = 0) \subset U_1$ and $(v = 0) \subset U_2$.

Then $\pi^{-1}\colon \mathbb{A}^2_{x,y} \dashrightarrow \mathrm{Bl}_0 \mathbb{A}^2_{x,y}$ is a rational map which is *not* a morphism.

Proposition 3.3. *Let X and Y be smooth projective curves and $f\colon X \to Y$ a morphism. Then either f is constant or f is surjective.*

Proof 1: Analysis ($k = \mathbb{C}$). Here we work in the analytic topology. f is locally of the form $z \mapsto w = z^e$ for some $e \in \mathbb{Z}$, $e \geq 1$. Here z and w are local coordinates at $P \in X$ and $Q = f(P) \in Y$. (Indeed, $z \mapsto w = f(z)$ where $f(z)$ is holomorphic, so $f(z) = a_e z^e + a_{e+1} z^{e+1} + \cdots$, where $a_e \neq 0$. Write $f(z) = (a_e + a_{e+1} z + \cdots) z^e = u \cdot z^e$. Since $u(P) \neq 0$ we can take an eth root $u^{1/e}$ in a neighbourhood of $P \in X$. Let $z' = u^{1/e} z$, then z' is a local coordinate at $P \in X$, and $z' \mapsto w = z'^e$.)

In particular f is open (if $U \subset X$ is open then $f(U) \subset Y$ is open). Now X is compact, so f is also closed. So $f(X)$ is open and closed, hence $f(X) = Y$. □

In the proof we used the following fact: a complex projective variety X is compact (in its analytic topology). To see this, observe first that complex projective space $\mathbb{P}^N_{\mathbb{C}}$ is compact. Indeed

$$\mathbb{P}^N_{\mathbb{C}} = (\mathbb{C}^{N+1} - \{0\})/\mathbb{C}^\times = S^{2N+1}/U(1)$$

where S^{2N+1} is the $(2N+1)$-sphere

$$(|z_0|^2 + \cdots + |z_N|^2 = 1) \subset \mathbb{C}^{N+1}$$

and

$$U(1) = (|\lambda| = 1) \subset \mathbb{C}^\times.$$

Now S^{2N+1} is compact and $\mathbb{P}^N$ is the image of S^{2N+1} under a continuous map, so $\mathbb{P}^N$ is compact. Finally, a complex projective variety is a closed subset of projective space so is also compact.

Proof 2: Algebra. Let $K \subset L$ be the corresponding extension of function fields. Given a point $P \in Y$ consider the corresponding DVR $A = \mathcal{O}_{Y,P} \subset K = k(Y)$. Let $B_0 \subset L$ be the integral closure of A in L. Localise at a maximal ideal of B_0 to obtain a DVR $B \subset L = k(X)$. Then $B = \mathcal{O}_{X,Q}$, some $Q \in X$, and $Q \mapsto P$. (Remark: Geometrically, $B_0 = \bigcap_{Q \in f^{-1}P} \mathcal{O}_{X,Q}$, the rational functions on X which are regular at every preimage of P. It is finite over $\mathcal{O}_{Y,P}$.) □

Remark 3.4. In general, if $f\colon X \to Y$ is a morphism of varieties, then $f(X) \subset Y$ is a *constructible set*, that is, a finite union of locally closed subsets. (Recall, a *locally closed* subset of a topological space is an open subset of a closed subset). See [Hartshorne, Ch. II, Ex 3.18, 3.19]. For example if

$$f\colon \mathbb{A}^2_{x,y} \to \mathbb{A}^2_{z,w}, \quad (x,y) \mapsto (z,w) = (x,xy),$$

then

$$f(\mathbb{A}^2_{x,y}) = (z \neq 0) \cup \{0\} \subset \mathbb{A}^2_{z,w}.$$

Remark 3.5. There is a notion of compactness in algebraic geometry called properness. Again the problem is that the usual definition of compactness does not work because the Zariski topology is too coarse (cf. Hausdorff/separated condition). We say an algebraic variety X is *proper* if for any variety Y the projection $\mathrm{pr}_2\colon X \times Y \to Y$ is closed (that is, if $Z \subset X \times Y$ closed then $\mathrm{pr}_2(Z) \subset Y$ is closed). (Note: this property characterises compact topological spaces among (reasonable) Hausdorff topological spaces). One shows that (1) projective space $\mathbb{P}^N$ is proper [Mumford, p. 104, Thm. 1], (2) if X is proper and $Y \subset X$ is closed then Y is proper, (3) if X is proper and $f\colon X \to Y$ is a morphism then $f(X) \subset Y$ is closed and proper. (Note: (1) is fairly hard, (2) and (3) follow easily from the definition.) In particular, a projective variety is proper, and the image of a projective variety under a morphism is closed (and proper). Cf. 3.4. If $k = \mathbb{C}$, then X is proper iff X^{an} is compact [Mumford, p. 114, Thm. 2].

For $f\colon X \to Y$ a morphism of smooth curves and $P \in X$ a point, we define the *ramification index* $e = e_P$ of f at P as follows:

($k = \mathbb{C}$): Locally at $P \in X$ and $Q = f(P) \in Y$, f is of the form $z \mapsto z^e$ (see proof 1 of Prop. 3.3).

(k arbitrary): Let $f^\sharp\colon \mathcal{O}_{Y,Q} \subset \mathcal{O}_{X,P}$ be the inclusion given by pullback of functions via f. Let w, z be local parameters at $Q \in Y$ and $P \in X$, and write $w = u \cdot z^e$, where $u \in \mathcal{O}_{X,P}$ is a unit (i.e., $u(P) \neq 0$).

Thus $e_P \in \mathbb{Z}$, $e_P \geq 1$. We say P is a *ramification point* of f if $e_P > 1$. We say $Q \in Y$ is a *branch point* of f if f is ramified at some pre-image of Q.

Proposition 3.6. *Let $f\colon X \to Y$ be a morphism of smooth projective curves. Then $\sum_{P \in f^{-1}Q} e_P$ is independent of $Q \in Y$ and is called the* degree *of f. We have $\deg(f) = [k(X) : k(Y)] = \dim_{k(Y)} k(X)$, the degree of the field extension $k(Y) \subset k(X)$.*

Proof 1: Analysis ($k = \mathbb{C}$). We observe that the sum $\sum_{P \in f^{-1}Q} e_P$ is a locally constant function of $Q \in Y$. Hence it is constant (since Y is connected). □

Proof 2: Algebra. (See [Hartshorne, II.6.9] for more details). Let $K = k(Y) \subset L = k(X)$, $A = \mathcal{O}_{Y,Q}$, and $B = \bigcap_{P \in f^{-1}Q} \mathcal{O}_{X,P}$. (So B is the integral closure of A in L). B is a finite A-module, A is a DVR, and B is torsion-free (that is, if $a \in A$, $b \in B$ and $ab = 0$, then $a = 0$ or $b = 0$). It follows that B is a free A-module, i.e., $B \simeq A^{\oplus r}$ as A-modules, for some r. Now we compute the rank r in 2 ways:

$$r = \mathrm{rk}(B \otimes_A K / A \otimes_A K) = \mathrm{rk}(L/K) = [L : K]$$

and

$$r = \mathrm{rk}(B \otimes_A k / A \otimes_A k) = \dim_k \left(\bigoplus_{P \in f^{-1}Q} \mathcal{O}_{X,P} / m_{Y,Q} \cdot \mathcal{O}_{X,P} \right)$$

$$= \sum_{P \in f^{-1}Q} \dim_k (\mathcal{O}_{X,P} / m_{X,P}^{e_P}) = \sum_{P \in f^{-1}Q} e_P.$$

□

4 Riemann–Hurwitz

Let X be a smooth projective curve over $k = \mathbb{C}$. Then the associated analytic space X^{an} is a compact complex manifold of (complex) dimension 1. The underlying smooth manifold is a compact oriented smooth manifold of (real) dimension 2. Hence it is homeomorphic to a sphere with g handles for some g, the *genus* of X.

If X is a topological space, such that X is homeomorphic to a finite union of simplices (a 'simplicial complex') we can define the *homology groups* of X as follows. Let $C_i(X, \mathbb{Z})$ be the free abelian group generated by the simplices of dimension i. Let $d\colon C_i(X, \mathbb{Z}) \to C_{i-1}(X, \mathbb{Z})$ be the map which sends a simplex σ to its boundary $\partial\sigma$. (A note about signs: for each simplex σ, we fix an orientation. We agree that $-\sigma$ corresponds to σ with the opposite orientation. Then the boundary $\partial\sigma$ of σ is by definition the sum of the codimension 1 faces of σ with the induced orientations.) One can check that $d^2 = 0$. We define the *i-th homology group* $H_i(X, \mathbb{Z})$ of X by

$$H_i(X, \mathbb{Z}) = \frac{\ker(d\colon C_i(X, \mathbb{Z}) \to C_{i-1}(X, \mathbb{Z}))}{\mathrm{im}(d\colon C_{i+1}(X, \mathbb{Z}) \to C_i(X, \mathbb{Z}))}$$

Call an element of $C_i(X, \mathbb{Z})$ an i-chain — this is just a formal sum of simplices of dimension i. Then, an element of $H_i(X, \mathbb{Z})$ corresponds to an i-chain which has no boundary (an i-cycle) modulo boundaries of $(i+1)$-chains. One can show that the abelian groups $H_i(X,\mathbb{Z})$ are determined by the topological space X (they do not depend on the triangulation, that is, the choice of realisation of X as a simplicial complex). The *Euler characteristic* of X is the number

$$e(X) = \sum_i (-1)^i \operatorname{rk} H_i(X,\mathbb{Z}).$$

For any triangulation, $e(X) = \sum_i (-1)^i N_i$, where N_i is the number of simplices of dimension i (this follows by linear algebra from the definition of the $H_i(X,\mathbb{Z})$ above).

If X is a sphere with g handles, then $H_0(X,\mathbb{Z}) = \mathbb{Z}$, $H_1(X,\mathbb{Z}) \simeq \mathbb{Z}^{2g}$, and $H_2(X,\mathbb{Z}) = \mathbb{Z}$. In particular, $e(X) = 2 - 2g$.

Theorem 4.1. *(Riemann–Hurwitz) $k = \mathbb{C}$. Let $f\colon X \to Y$ be a morphism of smooth projective curves of degree d. Then*

$$2g(X) - 2 = d(2g(Y) - 2) + \sum_{P\in X} (e_P - 1)$$

Proof. Pick a triangulation of Y that includes as vertices the branch points of f. (To visualise the argument, it may be helpful to think of the triangles as small, but this is not strictly necessary). The inverse images of the triangles in Y define a triangulation of X. Let V, E, F denote the number of vertices, edges, and faces of the triangulations. Then $F_X = dF_Y$, $E_X = dE_Y$ and $V_X = dV_Y - \sum_{P\in X}(e_P - 1)$. Here the correction term in the V_X formula accounts for the ramification over branch points of f. Recall that the Euler characteristic $e = V - E + F$. Hence $e(X) = d\cdot e(Y) - \sum_{P\in X}(e_P - 1)$. Now using $e = 2 - 2g$ gives the Riemann–Hurwitz formula. □

Remark 4.2. We define the genus of a smooth projective curve X over an arbitrary algebraically closed field k in Sec. 21. (For example, g is the dimension of the k-vector space of regular algebraic 1-forms on X). Using this, one can prove the Riemann–Hurwitz formula for arbitrary k under the assumption that the ramification of f is *tame*, that is, the ramification index e_P is not divisible by the characteristic of k for all $P \in X$.

5 Hyperelliptic curves

Let X be a smooth projective curve. We say X is *hyperelliptic* if there exists a morphism $X \to \mathbb{P}^1$ of degree 2. (Also, we usually assume that $g(X) \neq 0$, equivalently, X is not isomorphic to $\mathbb{P}^1$).

We give a complete classification of hyperelliptic curves. We assume that the characteristic of k is not equal to 2. A morphism $X \to \mathbb{P}^1$ of degree 2 corresponds to a field extension $k(\mathbb{P}^1) \subset k(X)$ of degree 2. Now $k(\mathbb{P}^1) = k(x)$ where $x = X_1/X_0$ is an affine coordinate on $\mathbb{P}^1$ and $k(x)$ denotes the field of rational functions in the indeterminate x. Since $\mathrm{char}(k) \neq 2$, we can complete the square and write $k(X) = k(x)(\sqrt{f})$ for some $f = f(x) \in k(x)$. Moreover we may assume that $f \in k[x]$ is a polynomial with distinct roots.

Let U be the affine curve

$$U = (y^2 = f(x)) \subset \mathbb{A}^2_{x,y}$$

Then U is a smooth affine curve with function field $k(U) = k(X)$, and the map $U \to \mathbb{A}^1_x$ is the restriction of the cover $X \to \mathbb{P}^1$ to $\mathbb{A}^1_x = (X_0 \neq 0) \subset \mathbb{P}^1$. We now describe the affine patch of X over $\mathbb{A}^1_z = (X_1 \neq 0) \subset \mathbb{P}^1$, where $z = x^{-1} = X_0/X_1$.

Write $f(x) = (x - \alpha_1) \cdots (x - \alpha_k)$ where the α_i are distinct. Then

$$y^2 = f(x) = z^{-k}(1 - \alpha_1 z) \cdots (1 - \alpha_k z).$$

Let $l = \lceil k/2 \rceil$, that is, $l = k/2$ if k even and $l = (k+1)/2$ if k odd. Then

$$(z^l y)^2 = g(z) := z^\delta (1 - \alpha_1 z) \cdots (1 - \alpha_k z).$$

where $\delta = 0$ or 1 as k is even or odd. Let $w = z^l y = x^{-l} y$. Then

$$V = (w^2 = g(z)) \subset \mathbb{A}^2_{z,w}$$

is a smooth affine curve, and $V \to \mathbb{A}^1_z$ is the restriction of $X \to \mathbb{P}^1$ to $\mathbb{A}^1_z \subset \mathbb{P}^1$. The glueing $X = U \cup V$ is given by

$$U \supset (x \neq 0) \xrightarrow{\sim} (z \neq 0) \subset V, \quad (x, y) \mapsto (z, w) = (x^{-1}, x^{-l} y).$$

The ramification points of the map $X \to \mathbb{P}^1$ are $(1 : \alpha_1), \ldots, (1 : \alpha_k)$, and $\infty = (0 : 1)$ if k is odd. (Note each ramification point necessarily has ramification index 2 because the map has degree 2). By Riemann–Hurwitz,

$$2g(X) - 2 = 2(-2) + \sum (e_P - 1) = -4 + r$$

where r is the number of ramification points. So $r = 2g(X) + 2$.

Remark 5.1. In particular, there are hyperelliptic curves of every genus $g \geq 0$. However, not every curve is hyperelliptic. For example, a smooth plane curve of degree $d \geq 4$ is never hyperelliptic by Prop. 23.3. More abstractly, the dimension of the moduli space of hyperelliptic curves of genus $g \geq 2$ has dimension $2g - 1$, whereas the moduli space of all smooth curves of genus $g \geq 2$ has dimension $3g - 3$. Now $2g - 1 < 3g - 3$ for $g \geq 3$, so there are nonhyperelliptic curves of genus g for $g \geq 3$. (The *moduli space* of hyperelliptic curves of genus g is the space parametrising such curves. One can compute its dimension using the above description: A hyperelliptic curve is determined by $2g+2$ points in $\mathbb{P}^1$ (the branch points) as above. Moreover, if $g \geq 2$, a hyperelliptic curve admits exactly one map $X \to \mathbb{P}^1$ of degree 2, up to composing with automorphisms of $\mathbb{P}^1$ (see Proof of Thm. 21.8). It follows that hyperelliptic curves of genus $g \geq 2$ correspond to sets of $2g + 2$ distinct points modulo automorphisms of $\mathbb{P}^1$. This parameter space has dimension $2g + 2 - \dim \mathrm{Aut}(\mathbb{P}^1) = 2g + 2 - \dim \mathrm{PGL}(2) = 2g - 1$. It is harder to compute the dimension of the moduli space of all smooth curves of genus g.)

6 Genus of plane curve

Corollary 6.1. $k = \mathbb{C}$. *Let* $X \subset \mathbb{P}^2$ *be a smooth plane curve of degree* d. *Then* $g(X) = \frac{1}{2}(d-1)(d-2)$.

Proof. If $P \in \mathbb{P}^2$ is a point not lying on X, the projection $\mathbb{P}^2 \dashrightarrow \mathbb{P}^1$ from P defines a morphism $f \colon X \to \mathbb{P}^1$ of degree d. We compute the ramification of this map and apply Riemann–Hurwitz to compute $g(X)$.

Assume $(1 : 0 : 0) \notin X$ and consider the projection from $(1 : 0 : 0)$:

$$\mathbb{P}^2 \dashrightarrow \mathbb{P}^1, \quad (X_0 : X_1 : X_2) \mapsto (X_1 : X_2).$$

Let $g \colon X \to \mathbb{P}^1$ be the induced map. Let $F = F(X_0, X_1, X_2)$ be the homogeneous equation of $X \subset \mathbb{P}^2$. We show that g is ramified at $P \in X$ iff $\frac{\partial F}{\partial X_0}(P) = 0$, and $(e_P - 1)$ is equal to the intersection multiplicity of X and $(\frac{\partial F}{\partial X_0}(P) = 0) \subset \mathbb{P}^2$ at P. Let $Q = g(P)$. We may assume $Q \in (X_1 \neq 0) = \mathbb{A}^1_{x_2} \subset \mathbb{P}^1_{(X_1:X_2)}$, where $x_2 = X_2/X_1$. The affine piece $(X_1 \neq 0)$ of $X \subset \mathbb{P}^2$ is given by $(f(x_0, x_2) = 0) \subset \mathbb{A}^2_{x_0,x_2}$ where $x_0 = X_0/X_1$, $x_2 = X_2/X_1$, and $f(x_0, x_2) = F(x_0, 1, x_2)$. The projection $X \to \mathbb{P}^1$ is locally given by

$$(f(x_0, x_2) = 0) \subset \mathbb{A}^2_{x_0,x_2} \to \mathbb{A}^1_{x_2}, \quad (x_0, x_2) \mapsto x_2$$

Thus $X \to \mathbb{P}^1$ is ramified at P iff $\frac{\partial f}{\partial x_0}(P) = 0$. Equivalently $\frac{\partial F}{\partial X_0}(P) = 0$ (because $\frac{\partial f}{\partial x_0} = \frac{\partial F}{\partial X_0}(x_0, 1, x_2)$). In this case, $\frac{\partial f}{\partial x_2}(P) \neq 0$, since X is smooth by assumption. So, by the inverse function theorem, we can write $x_2 = h(x_0)$ locally at $P \in X$. Differentiating $f(x_0, h(x_0)) = 0$ gives

$$\frac{\partial f}{\partial x_0} + \frac{\partial f}{\partial x_2} \cdot h'(x_0) = 0.$$

So $\nu_P(h'(x_0)) = \nu_P(\frac{\partial f}{\partial x_0})$, that is, $(e_P - 1)$ equals the intersection multiplicity of X with the curve $(\frac{\partial F}{\partial X_0} = 0) \subset \mathbb{P}^2$ at P.

We deduce that $\sum_{P \in X}(e_P - 1) = (F = 0) \cdot (\frac{\partial F}{\partial X_0} = 0) = d(d-1)$ by Bézout's theorem [Hartshorne, I.7.8]. Now Riemann–Hurwitz gives

$$2g(X) - 2 = d(-2) + d(d-1),$$

and we can solve for $g(X)$. □

Remark 6.2. The first few values are $g = 0, 0, 1, 3, 6, 10, \ldots$ for $d = 1, 2, \ldots$. In particular, not every curve can be embedded in the plane. However, every smooth projective curve can be embedded in $\mathbb{P}^3$. Indeed, given a smooth projective curve $X \subset \mathbb{P}^N$ with $N \geq 3$, a dimension count shows that we can find a point $P \in \mathbb{P}^N$ which does not lie on any secant line or tangent line of X. (A *secant line* is a line meeting X in at least 2 distinct points). Then projection from P defines an embedding $X \subset \mathbb{P}^{N-1}$. By induction we obtain $X \subset \mathbb{P}^3$. See [Hartshorne, IV.3.6] for more details. Also, every smooth projective curve admits an immersion $X \to \mathbb{P}^2$. (Here an *immersion* is a morphism which, locally analytically on X, is an embedding. It is not necessarily injective however). The image $Y \subset \mathbb{P}^2$ of X is a curve with only nodes as singularities, that is, each singular point $Q \in Y$ is locally analytically isomorphic to $(xy = 0) \subset \mathbb{A}^2_{x,y}$ (equivalently, $\hat{\mathcal{O}}_{Y,Q} \simeq k[[x,y]]/(xy)$). See [Hartshorne, IV.3.11]. If $Y \subset \mathbb{P}^2$ is a plane curve of degree d with δ nodes, and $X \to Y$ is the normalisation of Y (the unique smooth projective curve birational to Y) then $g(X) = \frac{1}{2}(d-1)(d-2) - \delta$.

Remark 6.3. The genus formula for a plane curve is true for an arbitrary field k. It is proved using algebraic differential forms and the 'adjunction formula', see Prop. 23.1.

7 Quotients by finite groups

Let X be an algebraic variety and G a finite group acting on X. We construct the quotient X/G as an algebraic variety.

First, as a set X/G is the set of orbits of G. In other words, it is the set of equivalence classes for the equivalence relation $\sim$ defined by $x \sim y$ if

$y = gx$ for some $g \in G$. Second, as a topological space, we give X/G the quotient topology. That is, writing $q\colon X \to X/G$ for the quotient map, a set $U \subset X/G$ is open iff $q^{-1}U \subset X$ is open. (This is the finest topology on X/G such that q is continuous).

Now, we give X/G the structure of an algebraic variety. We first reduce to the affine case. Recall $X = \bigcup U_i$, a finite union of open affine subvarieties.

Lemma 7.1. *Assume that X is projective. Then X has a finite open covering by G-invariant open affines.*

Proof. Let $x \in X$ be a point. Since X is projective, we can find an open affine $U \subset X$ which contains gx for every $g \in G$. Indeed, let $X \subset \mathbb{P}^N$ be a projective embedding, and take a hyperplane $H \subset \mathbb{P}^N$ not containing any gx. Then $U := X \setminus X \cap H \subset \mathbb{P}^N \setminus H \simeq \mathbb{A}^N$ will do. Now let $V = \bigcap_{g \in G} gU$. Then V is a G-invariant open neighbourhood of x, and is affine by Lem. 7.2. Hence we get a open covering of X by G-invariant open affines. Finally, every open covering of a variety admits a finite open subcovering. $\square$

Lemma 7.2. *Let X be a variety and $U, V \subset X$ open affine subvarieties. Then $U \cap V$ is also affine.*

Proof. By the separatedness condition, the diagonal $\Delta \subset X \times X$ is closed. Indeed, the diagonal is the locus $(\mathrm{pr}_1 = \mathrm{pr}_2) \subset X \times X$ where

$$\mathrm{pr}_1, \mathrm{pr}_2 \colon X \times X \to X$$

are the two projections. (In fact $\Delta \subset X \times X$ is closed iff X is separated: if $f, g\colon Z \to X$ are two maps, then $(f = g) = (f, g)^{-1}\Delta$, so $(f = g) \subset Z$ is closed if $\Delta \subset X \times X$ is closed. This is sometimes taken as the definition of separatedness.)

Now observe $U \cap V \simeq U \times V \cap \Delta$. So $U \cap V$ is a closed subvariety of the affine variety $U \times V$, hence affine. $\square$

Now let X be an affine variety and G a finite group acting on X. The action of G on X corresponds to an action of G on the coordinate ring $k[X]$ (by the usual correspondence between morphisms of affine varieties and morphisms of their coordinate rings).

Notation 7.3. When a group G acts on a ring A we write

$$A^G = \{a \in A \mid ga = a \ \forall g \in G\}$$

for the ring of invariants.

Lemma 7.4. *The invariant ring $k[X]^G$ is a finitely generated k-algebra.*

Proof. The coordinate ring $k[X]$ is a finitely generated k-algebra. Pick generators $x_1, \ldots, x_N$. For each i, write

$$\prod_{g \in G} (T - gx_i) = T^n + a_{i,n-1}T^{n-1} + \ldots + a_{i,1}T + a_{i,0},$$

where $n = |G|$. Then $a_{ij} \in k[X]^G$ for all i, j by construction. Consider $A := k[a_{ij}] \subset k[X]^G$, the subalgebra generated by the a_{ij}. Then $A \subset k[X]$ is finite (i.e. $k[X]$ is a finitely generated A-module) because each generator x_i of $k[X]$ as a k-algebra satisfies a monic polynomial with coefficients in A. If A is a Noetherian ring, M is a finitely generated A-module, and $N \subset M$ is a submodule, then N is also finitely generated. Applying this to $k[X]^G \subset k[X]$ we deduce that $k[X]^G$ is a finitely generated A-module. So in particular $k[X]^G$ is a finitely generated k-algebra. $\square$

We define the variety $X/G = \operatorname{Spec} k[X]^G$.

(Recall that, for A a finitely-generated k-algebra, $\operatorname{Spec} A$ denotes the affine variety with coordinate ring A).

Lemma 7.5. *The underlying topological space of the variety X/G is the set of orbits with the quotient topology.*

Proof. We first show that the underlying set is the set of orbits. Since the inclusion of rings $k[X]^G \subset k[X]$ is finite (see the proof of Lem. 7.4) the corresponding morphism $q\colon X \to X/G$ is surjective and closed. So, to show that the underlying set of X/G is the set of orbits, it suffices to show that $q(x) = q(y)$ implies $y = gx$ for some $g \in G$. Suppose $x, y \in X$ and $y \neq gx$ for all $g \in G$. Pick $f \in k[X]$ such that $f(y) = 0$ and $f(gx) \neq 0$ for all $g \in G$. (For example, take f to be the equation of a suitable affine hyperplane in some embedding $X \subset \mathbb{A}^N$). Let $h = \prod_{g \in G} gf \in k[X]^G$. Then $h(y) = 0$ and $h(x) \neq 0$, so $q(x) \neq q(y)$.

Second, we show that the topology on X/G is the quotient topology. Suppose $U \subset X/G$ a subset such that $q^{-1}U$ is open. We need to show that U is open. This holds because q is closed and $q(X \setminus q^{-1}U) = (X/G) \setminus U$. $\square$

It only remains to glue the affine pieces together. Given G-invariant open affine subvarieties $U, V \subset k[X]$, the glueing of U and V along $U \cap V$ is given by the inclusions $k[U], k[V] \subset k[U \cap V]$ (recall that $U \cap V$ is affine). Taking G-invariants gives the glueing of U/G and V/G along $U \cap V/G$.

Example 7.6. Let $X = \mathbb{A}^2_{x,y}$ and $G = \mathbb{Z}/2\mathbb{Z}$ acting via $(x, y) \mapsto (-x, -y)$. Then

$$X/G = \operatorname{Spec} k[x, y]^G = \operatorname{Spec} k[x^2, xy, y^2] = \operatorname{Spec} k[u, v, w]/(uw = v^2)$$

That is,

$$X/G = (uw = v^2) \subset \mathbb{A}^3_{u,v,w}$$

a quadric cone.

8 Local theory of finite quotients

Let G be a finite group acting on a variety X. Let $q \colon X \to X/G$ be the quotient, and write $Y = X/G$. Let $P \in X$ be a point and $Q = q(P) \in Y$. Let $G_P \subset G$ be the stabiliser of $P \in X$.

Lemma 8.1. $\hat{\mathcal{O}}_{Y,Q} = \hat{\mathcal{O}}^{G_P}_{X,P}$.

Proof. We may assume that X is affine. The morphism q corresponds to the finite inclusion $k[X]^G \subset k[X]$. Localising at $Q \in Y$ we obtain $\mathcal{O}_{Y,Q} \subset \bigcap_{P \in q^{-1}Q} \mathcal{O}_{X,P}$. Completing at $Q \in Y$ gives $\hat{\mathcal{O}}_{Y,Q} \subset \bigoplus_{P \in q^{-1}Q} \hat{\mathcal{O}}_{X,P}$. Now

$$\mathcal{O}_{Y,Q} = \left(\bigoplus_{P \in q^{-1}Q} \hat{\mathcal{O}}_{X,P} \right)^G \simeq \hat{\mathcal{O}}^{G_P}_{X,P}$$

where P is any pre-image of Q. ☐

Lemma 8.2. *Let X be an algebraic variety of dimension n and $P \in X$ a point. Then $P \in X$ is smooth iff $\hat{\mathcal{O}}_{X,P} \simeq k[[x_1, \dots, x_n]]$.*

Proof. Recall that $P \in X$ is smooth iff $\mathcal{O}_{X,P}$ is a regular local ring, that is, the maximal ideal $m_{X,P} \subset \mathcal{O}_{X,P}$ is generated by $n = \dim X$ elements. Suppose first $P \in X$ is smooth. Pick generators $x_1, \dots, x_n$ of $m_{X,P}$, and consider the map

$$k[[X_1, \dots, X_n]] \to \hat{\mathcal{O}}_{X,P}, \quad X_i \mapsto x_i.$$

This map is an isomorphism (exercise).

Conversely, if $\hat{\mathcal{O}}_{X,P} \simeq k[[x_1, \dots, x_n]]$, then $m_{X,P}/m^2_{X,P}$ has dimension n, and so $m_{X,P}$ is generated by n elements (by Nakayama's lemma). ☐

Lemma 8.3. *Let G be a finite group acting on $k[[x_1, \ldots, x_n]]$ Assume that $(|G|, \operatorname{char} k) = 1$. Then, after a change of coordinates, we may assume that G acts linearly.*

Proof. Let $\rho\colon G \to \mathrm{GL}_n(k)$ be the representation of G which is the linearisation of the action of G on $k[[x_1, \ldots, x_n]]$. That is, write $gx_i = \sum a_{ji}x_j + \cdots$, where $\cdots$ denote higher order terms, then $\rho(g) = (a_{ij}) \in GL_n(k)$, (So, if $k = \mathbb{C}$, $\rho(g)$ is the dual of the derivative of $g\colon (0 \in \mathbb{C}^n) \to (0 \in \mathbb{C}^n)$ at $0 \in \mathbb{C}^n$). We also write $\rho(g)$ for the corresponding automorphism of $k[[x_1, \ldots, x_n]]$ given by $x_i \mapsto \sum a_{ji}x_j$.

Define an automorphism

$$\theta\colon k[[x_1, \ldots, x_n]] \to k[[x_1, \ldots, x_n]]$$

by

$$x_i \mapsto \frac{1}{|G|} \sum_{g \in G} g\rho(g)^{-1} x_i$$

We claim that $\rho(g) = \theta^{-1} g \theta$ for all $g \in G$. That is, in the new coordinates given by θ, the action of $g \in G$ is given by the linear map $\rho(g)$. Equivalently, $g\theta = \theta\rho(g)$. It suffices to check this on the x_i. We compute

$$g\theta(x_i) = \frac{1}{|G|} \sum_{h \in G} gh\rho(h)^{-1} x_i = \frac{1}{|G|} \sum_{h' \in G} h'\rho(h')^{-1}\rho(g) x_i = \theta\rho(g) x_i$$

(here we used the substitution $h' = gh$). □

9 Quotients of curves

Let G be a finite group acting on a smooth curve X. We assume (without loss of generality) that G acts faithfully on X (that is, if $gx = x$ for all $x \in X$ then $g \in G$ is the identity element).

Proposition 9.1. *Assume $(|G|, \operatorname{char} k) = 1$. The quotient X/G is smooth. The quotient map $q\colon X \to X/G$ is finite of degree $|G|$. The ramification index at a point $P \in X$ equals the order of the stabiliser $G_P \subset G$ of $P \in X$.*

Proof. Let $P \in X$ and $Q = q(P) \in Y = X/G$. We can choose an isomorphism $\mathcal{O}_{X,P} \simeq k[[x]]$ such that G_P acts linearly via $\rho\colon G_P \to \mathrm{GL}_1(k) = k^\times$. G_P acts faithfully on $\mathcal{O}_{X,P}$ (because G acts faithfully on X by assumption).

So ρ is injective and G_P is identified with the group μ_e of eth roots of unity in $k^\times$ for some e. The action is $\mu_e \ni \zeta \colon x \mapsto \zeta x$. Thus

$$\hat{\mathcal{O}}_{Y,Q} = \mathcal{O}_{X,P}^{G_P} = k[[x]]^{\mu_e} = k[[x^e]] = k[[y]].$$

In particular $Q \in Y$ is smooth. Also $(P \in X) \to (Q \in Y)$ is given by $x \mapsto y = x^e$, so $e = |G_P|$ is the ramification index of q at $P \in X$.

The quotient map $q : X \to X/G$ is a finite map: locally on X/G it is given by a finite inclusion of rings $k[U]^G \subset k[U]$ (where $U \subset X$ is a G-invariant open affine). The degree of q is the cardinality of a general fibre, and the fibres of q are the G-orbits, so $\deg q = |G|$. □

Proposition 9.2. *($k = \mathbb{C}$) Let X be a smooth projective curve, G a finite group acting on X, and $q \colon X \to Y := X/G$ the quotient. For each $Q \in Y$ let e_Q denote the common ramification index of q at the points $P \in q^{-1}Q$. Then*

$$2g(X) - 2 = |G| \left(2g(Y) - 2 + \sum_{Q \in Y} \left(1 - \frac{1}{e_Q} \right) \right)$$

Proof. We use the Riemann–Hurwitz formula

$$2g(X) - 2 = \deg q \cdot (2g(Y) - 2) + \sum_{P \in X} (e_P - 1).$$

Recall that $\deg q = |G|$ and the ramification index e_P at a point $P \in X$ is the order of the stabiliser G_P. So, over a branch point $Q \in Y$ we have $|G|/e_Q$ points with ramification index e_Q by the orbit-stabiliser theorem. This gives the above formula. □

Theorem 9.3. *($k = \mathbb{C}$) Let X be a smooth projective curve of genus $g(X) \geq 2$. Let G be a finite group acting on X. Then $|G| \leq 84(g(X) - 1)$.*

Proof. We use the result and notation of 9.2. Let $R = \sum_{Q \in Y} (1 - \frac{1}{e_Q})$. Then

$$0 < 2g(X) - 2 = |G|(2g(Y) - 2 + R).$$

Suppose first $g(Y) \geq 1$. If $R = 0$ then $g(Y) \geq 2$ and $|G| \leq (g(X) - 1)$. If $R \neq 0$ then $R \geq \frac{1}{2}$, so $2g(Y) - 2 + R \geq \frac{1}{2}$ and $|G| \leq 4(g(X) - 1)$.

Now suppose $g(Y) = 0$. Then $0 < 2g(X) - 2 = |G|(R - 2)$, so $R > 2$. By checking cases we find $R \geq 2\frac{1}{42}$ (with equality iff $\{e_Q\} = \{2, 3, 7\}$). So $|G| \leq 84(g(X) - 1)$. □

Remark 9.4. In fact a curve X of genus $g \geq 2$ has finite automorphism group (see Thm. 21.8), so $|\operatorname{Aut}(X)| < 84(g-1)$.

Remark 9.5. If $g(X) \leq 1$ then $\operatorname{Aut} X$ is infinite. Indeed, if $g(X) = 0$ then $X \simeq \mathbb{P}^1$ and $\operatorname{Aut}(X) = \mathrm{PGL}(2, \mathbb{C})$ (Möbius transformations). If $g(X) = 1$ then the Riemann surface X^{an} is a *complex torus* $\mathbb{C}/\Lambda$. Here $\Lambda \subset \mathbb{C}$ is a lattice, that is, Λ is a free abelian group of rank 2 which generates $\mathbb{C}$ as an $\mathbb{R}$-vector space. There is an exact sequence of groups

$$0 \to X \to \operatorname{Aut}(X) \to G \to 0$$

where X acts on itself by translations and G is a finite group. Explicitly, G is the group of automorphisms of $\mathbb{C}/\Lambda$ fixing the origin. These are given by multiplication by a scalar $0 \neq c \in \mathbb{C}$ such that $c\Lambda = \Lambda$. Thus $G = \mathbb{Z}/4\mathbb{Z}$ for the square lattice $\mathbb{Z} \oplus \mathbb{Z}i$, $\mathbb{Z}/6\mathbb{Z}$ for the hexagonal lattice $\mathbb{Z} \oplus \mathbb{Z}\omega$, and $\mathbb{Z}/2\mathbb{Z}$ otherwise.

Example 9.6. Let $X = (X^3Y + Y^3Z + Z^3X = 0) \subset \mathbb{P}^2_{(X:Y:Z)}$, the *Klein quartic*. Then X is a smooth plane quartic curve, so $g(X) = 3$. One can show that $|\operatorname{Aut}(X)| = 84(g(X) - 1) = 168$. In fact $\operatorname{Aut}(X) \simeq \mathrm{PGL}(2, \mathbb{F}_7)$, the simple group of order 168. See [Thurston].

10 Covers of curves via monodromy representations

Theorem 10.1. *($k = \mathbb{C}$) Let Y be a smooth projective curve. Let $B \subset Y$ be a finite subset. Fix a base point $q \in Y$. There is a bijection between the following sets:*

(1) Degree d morphisms $f \colon X \to Y$ from a smooth projective curve X with branch points contained in B, modulo isomorphism.

(2) Group homomorphisms $\rho \colon \pi_1(Y - B, q) \to S_d$ with image a transitive subgroup of S_d, modulo conjugation by an element of S_d.

Moreover, for $b \in B$ and γ a small loop in $Y \setminus B$ around b and based at q, $\rho(\gamma)$ has cycle type $(e_1, \ldots, e_k)$ iff there are k points in $f^{-1}b$ with ramification indices $e_1, \ldots, e_k$.

Here, an *isomorphism* of two covers $f \colon X \to Y$ and $f' \colon X' \to Y$ of Y is an isomorphism $g \colon X \to X'$ such that $f'g = f$. By a *small loop γ around b based at q* we mean the following: take q' close to b and a small loop γ'

around b based at q' (in the obvious sense), and α a path from q to q', and let $\gamma = \alpha\gamma'\alpha^{-1}$ (concatenation of paths). S_d denotes the symmetric group on d letters. An element σ of S_d decomposes as a disjoint union of cycles. The *cycle type* of σ is the sequence $(e_1, \ldots, e_k)$ of cycle lengths (a partition of d), and corresponds to the conjugacy class of $\sigma \in S_d$.

Proof. First we construct ρ given f. Write $Y^0 = Y \setminus B$, $X^0 = f^{-1}Y^0$, and let $f^0 \colon X^0 \to Y^0$ be the restriction of f. Then f^0 is a covering map (for the analytic topology), and every point $p \in Y^0$ has exactly d pre-images. Recall that a *covering map* is a map $g \colon Z \to W$ such that Z is connected, g is surjective, and, for all $p \in W$, there is an open neighbourhood $p \in U \subset W$ such that $g^{-1}U$ is a disjoint union $\coprod_{i \in I} V_i$ of open subsets $V_i \subset Z$ with $V_i \to U$ a homeomorphism. Let $f^{-1}(q) = \{p_1, \ldots, p_d\}$. Covering maps $g \colon Z \to W$ have the *path lifting property*: given a path $\gamma \colon [0,1] \to W$, there exists $\tilde{\gamma} \colon [0,1] \to Z$ such that $g\tilde{\gamma} = \gamma$, and $\tilde{\gamma}$ is uniquely determined if we fix $\tilde{\gamma}(0)$. Let γ be a loop in Y^0 based at q, and $\tilde{\gamma}_i$ the lift of γ such that $\tilde{\gamma}_i(0) = p_i$. We define $\rho(\gamma) \in S_d$ by $\rho(\gamma)^{-1}(i) = j$ if $\tilde{\gamma}_i(1) = p_j$. One checks that this gives a well defined homomorphism $\rho \colon \pi_1(Y^0, q) \to S_d$, the *monodromy representation* of f^0. Changing the labelling of $f^{-1}(q)$ corresponds to composing ρ with conjugation by an element of S_d.

For $b \in B$ a branch point and γ a small loop around b based at q, we can describe $\rho(\gamma)$ as follows. Locally over $b \in Y$, X is a disjoint union of k discs $\Delta = (|z| < 1) \subset \mathbb{C}$ and the map $X \to Y$ is given by $z \mapsto z^{e_i}$ on the ith disc, where e_i is the ramification index. Now it is easy to see that a small loop γ' around $b \in Y$ based at a point q' near b defines a permutation of the fibre $f^{-1}q'$ which is a product of k distinct cycles of lengths $e_1, \ldots, e_k$. Finally, replacing γ' by a loop $\gamma = \alpha\gamma'\alpha^{-1}$ based at q as above just corresponds to picking an identification of $f^{-1}(q)$ with $f^{-1}(q')$.

Now suppose given ρ, we construct f. First, we construct $f^0 \colon X^0 \to Y^0$ as a map of topological spaces. Recall that the *universal covering* of a topological space Z is a covering map $\pi \colon \tilde{Z} \to Z$ such that $\tilde{Z}$ is simply connected. If $p \colon W \to Z$ is a covering map then π factors through p, that is, there is a covering map $g \colon \tilde{Z} \to W$ such that $\pi = pg$. The fundamental group $\pi_1(Z, z)$ acts on the universal covering $\pi \colon \tilde{Z} \to Z$, and the action is simply transitive on fibres of π, so that $Z = \tilde{Z}/\pi_1(Z, z)$. Covering maps $p \colon W \to Z$ correspond to conjugacy classes of subgroups $H \subset \pi_1(Z, z)$ as follows: given p, pick a basepoint $w \in p^{-1}(z)$ and define $H = p_*\pi_1(W, w) \subset \pi_1(Z, z)$; given H, let p be the covering space $\tilde{Z}/H \to Z$. Now let $\tilde{Y}^0 \to Y^0$ be the universal covering. Let X^0 be the quotient of $\tilde{Y}^0 \times \{1, \ldots, d\}$ by $\pi_1(Y^0, q)$, where $\pi_1(Y^0, q)$ acts on the universal covering $\tilde{Y}^0 \to Y^0$ as above and on

$\{1, \ldots, d\}$ by $\rho \colon \pi_1(Y^0, q) \to S_d$. Then the quotient of the projection

$$\tilde{Y}^0 \times \{1, \ldots, d\} \to \tilde{Y}^0$$

is a covering map $f^0 \colon X^0 \to Y^0$ of degree d with monodromy representation ρ. See [Hatcher, pp. 68–70].

Note that, since f^0 is a covering map and Y^0 a Riemann surface, there is a unique structure of a Riemann surface on X^0 such that f^0 is holomorphic. So we obtain $f^0 \colon X^0 \to Y^0$, a morphism of Riemann surfaces.

It remains to "plug the holes", that is, extend f^0 to a morphism of compact Riemann surfaces $f \colon X \to Y$. For $b \in B$, consider a small disc Δ at $b \in Y$. The covering map $f^0 \colon X^0 \to Y^0$ restricts to a disjoint union of covering spaces of the punctured disc $\Delta^\times = (0 < |z| < 1) \subset \mathbb{C}$. Observe that finite covering spaces of $\Delta^\times$ are given by

$$\Delta^\times \to \Delta^\times, \quad z \mapsto z^m$$

for some $m \in \mathbb{Z}$, $m \geq 1$. Indeed, the universal covering of $\Delta^\times$ is the upper half plane $\mathcal{H} = (\operatorname{Im} z > 0) \subset \mathbb{C}$ with covering map

$$\pi \colon \mathcal{H} \to \Delta^\times, \quad z \mapsto e^{2\pi i z}.$$

The action of $\pi_1(\Delta^\times, q) \simeq \mathbb{Z}$ on $\mathcal{H}$ is given by

$$\mathbb{Z} \ni n \colon z \mapsto z + n.$$

The covering maps of finite degree are given by quotients of $\mathcal{H}$ by subgroups of $\pi_1(\Delta^\times, q)$ of finite index, that is, $m\mathbb{Z} \subset \mathbb{Z}$ for some $m \geq 1$. Now $\mathcal{H}/m\mathbb{Z} = \Delta^\times$ with coordinate $w = e^{2\pi i z/m}$, so the covering map $\mathcal{H}/m\mathbb{Z} \to \mathcal{H}/\mathbb{Z}$ is the map $\Delta^\times \to \Delta^\times$, $w \mapsto w^m$. It follows that we can glue in discs over $B \subset Y$ to obtain a compact Riemann surface X and a morphism $f \colon X \to Y$.

Given a smooth complex projective variety X, we can consider the associated compact complex manifold X^{an}. It is a (hard) theorem that every compact Riemann surface is associated to a unique smooth projective curve, and morphisms of compact Riemann surfaces correspond to morphisms of projective curves. In particular the morphism $f \colon X \to Y$ of Riemann surfaces constructed above corresponds to a morphism of projective curves. (In our situation, since we have a finite morphism $f \colon X \to Y$, and Y is assumed algebraic, we can use a weaker result, the "Riemann existence theorem", to conclude that X is algebraic. See [Hartshorne, App. B, Thm. 3.2]) $\square$

11 The fundamental group of a Riemann surface

We give a presentation for the fundamental group of a punctured Riemann surface. We use it to explicitly describe coverings of Riemann surfaces via monodromy representations.

Theorem 11.1. *Let Y be a compact Riemann surface of genus g. Let $B = \{p_1, \dots, p_n\} \subset Y$ be a finite subset. Fix a basepoint $q \in Y \setminus B$. Then*

$$\pi_1(Y) \simeq \frac{\langle a_1, b_1, \dots, a_g, b_g \rangle}{([a_1, b_1] \cdots [a_g, b_g] c_1 \cdots c_n)}$$

Here $[a, b]$ denotes the commutator $aba^{-1}b^{-1}$.

In particular, if $n \geq 1$, then $\pi_1(Y)$ is a free group on $2g+n-1$ generators (because we can use the relation to eliminate c_n).

Proof. A compact Riemann surface Y of genus g can be obtained (as a topological space) from a polygon P with $4g$ edges as follows. Going anticlockwise around the boundary of the polygon we label the edges $a_1, b_1, a_1^{-1}, b_1^{-1}$, …, $a_g, b_g, a_g^{-1}, b_g^{-1}$. Now glue pairs of edges with the same label, respecting the orientations (here a^{-1} corresponds to a with the opposite orientation). Please draw a picture! All the vertices of the polygon are identified to a single point $q \in Y$ and the edges become loops a_i, b_i based at $q \in Y$. Using the Van Kampen theorem we compute that

$$\pi_1(Y, q) =< a_1, b_1, \dots, a_g, b_g > /([a_1, b_1] \dots [a_g, b_g]).$$

See [Fulton, 17c, p. 242]. Now consider the punctured surface $Y - \{p_1, \dots, p_n\}$. Let p be a vertex of the polygon P, and $c_1, \dots, c_n$ loops in $P \setminus \{p_1, \dots, p_n\}$ based at p such that c_i goes once clockwise around p_i and the concatenation $c_1 \cdots c_n$ is homotopy equivalent to the boundary of P traversed clockwise. Then a similar calculation shows that

$$\pi_1(Y - B, q) =< a_1, b_1, \dots, a_g, b_g > /([a_1, b_1] \dots [a_g, b_g] c_1 \cdots c_n).$$

□

Remark 11.2. It is a general fact that, for a cell complex X, the fundamental group $\pi_1(X, x)$ is the quotient of the fundamental group of the 1-skeleton of X by relations given by the 2-cells of X. See [Hatcher, Prop. 1.26, p. 50]. In our example $Y - \{p_1, \dots, p_n\}$ is (homotopy equivalent to) the cell complex with 1-cells the a_i, b_i, c_j and a unique 2-cell given by the part of the inteior of P lying outside the loops c_j. This gives the result above.

Example 11.3. If $g(Y) = 1$ then $\pi_1(Y, q) =< a, b > /([a, b]) \simeq \mathbb{Z}^2$.

If $Y = \mathbb{P}^1$ then

$$\pi_1(\mathbb{P}^1 - \{p_1, \ldots, p_n\}, q) \simeq < c_1, \ldots, c_n > /(c_1 \cdots c_n).$$

Thus, a degree d cover of $\mathbb{P}^1$ branched over the p_i corresponds to a set of permutations $\sigma_i \in S_d$ such that $\sigma_1 \cdots \sigma_n = 1$, modulo simultaneous conjugation by elements of S_d. Here σ_i is a small loop around p_i based at q, so the cycle type of σ_i gives the ramification indices over p_i.

Example 11.4. Let $f \colon \mathbb{P}^1 \to \mathbb{P}^1$ be the morphism given by $f = (1 : g)$ where

$$g(x) = 4x^2(x-1)^2/(2x-1)^2.$$

(Here $x = X_1/X_0$ is an affine coordinate on $\mathbb{P}^1$). Then f is a degree 4 morphism with 3 branch points, and monodromy representation given by $\sigma_1 = (12)(34)$, $\sigma_2 = (13)(24)$, $\sigma_3 = (14)(23)$ up to conjugacy (exercise).

Theorem 11.5. *($k = \mathbb{C}$) Let X be a smooth projective curve and G a finite group. Then there exists a Galois covering $Y \to X$ with group G, that is, a smooth projective curve Y with a G-action such that $X = Y/G$.*

Proof. Let $B = \{p_1, \ldots, p_n\} \subset X$ be a finite subset of size n and $q \in X \setminus B$ a basepoint. Write $X^0 = X - B$. Then $\pi_1(X^0, q)$ is a free group on $2g + n - 1$ generators (for $n \geq 1$). In particular, for $n \gg 0$, there exists a surjection $\pi_1(X^0, q) \twoheadrightarrow G$. This corresponds to a Galois covering of topological spaces $Y^0 \to X^0$ with group G. (Recall that covering maps $W \to Z$ correspond to subgroups $H \subset \pi_1(Z, z)$: given H, we define $W = \tilde{Z}/H \to Z$ where $\tilde{Z} \to Z$ is the universal cover. The covering $W \to Z$ is Galois iff H is normal, and in this case the Galois group is $G = \pi_1(Z, z)/H$.) Now Y^0 inherits the structure of a Riemann surface from X^0, and we can plug holes as before to obtain $Y \to X$. $\square$

We rephrase Thm. 11.5 in terms of algebra.

Theorem 11.6. *Let $\mathbb{C} \subset K$ be a finitely generated field extension of transcendence degree 1 and G a finite group. Then there exists a Galois extension $K \subset L$ with group G.*

Proof. $K = k(X)$ for some smooth projective curve X. Now take $Y \to X$ as in 11.5 and $L = k(Y)$. $\square$

Remark 11.7. The analogue of 11.6 for $K = \mathbb{Q}$ is not known.

12 Differential forms in algebraic geometry

We define the notion of a differential form on an algebraic variety.

12.1 Differential forms on smooth manifolds

First, we review the case of smooth manifolds. Let X be a smooth manifold of dimension n. The *tangent bundle* $TX \to X$ is a vector bundle of rank n. The fibre over a point $P \in X$ is T_PX, the *tangent space* to X at P.

The tangent bundle can be constructed using charts for X as follows. Let $X = \bigcup U_i$ be an open covering and $\phi_i \colon U_i \xrightarrow{\sim} V_i \subset \mathbb{R}^n$ charts. We identify $TV_i = V_i \times \mathbb{R}^n$ using $V_i \subset \mathbb{R}^n$. We glue the $TV_i \to V_i$ to obtain $TX \to X$ by

$$\begin{array}{ccc}
TV_i|_{\phi_i(U_i \cap U_j)} & \xrightarrow{D\psi_{ij}} & TV_j|_{\phi_j(U_i \cap U_j)} \\
{\scriptstyle \mathrm{pr}_1}\downarrow & & {\scriptstyle \mathrm{pr}_1}\downarrow \\
\phi_i(U_i \cap U_j) & \xrightarrow{\psi_{ij}} & \phi_j(U_i \cap U_j)
\end{array}$$

where $\psi_{ij} = \phi_j \circ \phi_i^{-1}$ and $D\psi_{ij}$ is the derivative of ψ_{ij}. That is,

$$D\psi_{ij}(P, v) = (\psi_{ij}(P), D\psi_{ij}(P) \cdot v)$$

where $D\psi_{ij}(P)$ is the matrix of first partial derivatives of ψ_{ij} evaluated at P.

Alternatively, we can construct the tangent bundle using derivations as follows. Let $C^\infty(X)$ denote the $\mathbb{R}$-algebra of smooth $\mathbb{R}$-valued functions on X. For $P \in X$, let $C^\infty(X)_P$ denote the associated local ring of germs of smooth $\mathbb{R}$-valued functions on a neighbourhood of $P \in X$. (That is, we consider smooth functions $f \colon U \to \mathbb{R}$ for U a neighbourhood of $P \in X$ modulo the equivalence relation $f \colon U \to \mathbb{R} \sim g \colon V \to \mathbb{R}$ if $f|_W = g|_W$ for some neighbourhood $W \subset U \cap V$ of $P \in X$.) We say $D \colon C^\infty(X) \to \mathbb{R}$ is a *derivation* if $D(fg) = f(P)Dg + g(P)Df$ (the "Leibniz rule"), $D(f+g) = Df + Dg$, and $D(\lambda) = 0$ for $\lambda \in \mathbb{R}$.

Lemma 12.1. *Let $x_1, \ldots, x_n$ be local coordinates at $P \in X$. The set of derivations $D \colon C^\infty(X)_P \to \mathbb{R}$ is an $\mathbb{R}$-vector space with basis $\{\frac{\partial}{\partial x_i}\}$, where*

$$\frac{\partial}{\partial x_i} \colon C^\infty(X)_P \to \mathbb{R}, \quad f \mapsto \frac{\partial f}{\partial x_i}(0).$$

Proof. Clearly the set of derivations is an $\mathbb{R}$-vector space and the $\frac{\partial}{\partial x_i}$ are linearly independent derivations. It remains to show that they span. If f is a smooth function on a neighbourhood of $P \in X$, then

$$\begin{aligned} f(x_1, \ldots, x_n) - f(0) &= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}\,t} f(tx_1, \ldots, tx_n)\,\mathrm{d}\,t \\ &= \sum_{i=1}^n x_i \left(\int_0^1 \frac{\partial f}{\partial x_i}(tx_1, \ldots, tx_n)\,\mathrm{d}\,t \right) = \sum_{i=1}^n x_i g_i \end{aligned}$$

where the g_i are smooth and $g_i(0) = \frac{\partial f}{\partial x_i}(0)$. So if $D\colon C^\infty(X)_P \to \mathbb{R}$ is a derivation then $Df = \sum_{i=1}^n (Dx_i)\frac{\partial f}{\partial x_i}(0)$ by the Leibniz rule. □

Thus we can define $T_P X$, the tangent space to X at P, as the $\mathbb{R}$-vector space of derivations $D\colon C^\infty(X)_P \to \mathbb{R}$.

Similarly, for $U \subset X$ an open set, we say a map $D\colon C^\infty(U) \to C^\infty(U)$ is a *derivation* if $D(fg) = gDf + fDg$, $D(f+g) = Df + Dg$, and $D\lambda = 0$ for $\lambda \in \mathbb{R}$. A derivation $D\colon C^\infty(U) \to C^\infty(U)$ corresponds to a smooth section of the tangent bundle $TX \to X$ over $U \subset X$ (that is, a smooth vector field on U).

A (smooth) differential k-form ω on X is a smooth section of $\wedge^k T^*X \to X$, the kth exterior power of the dual of the tangent bundle. (In coordinates: if $U \subset X$ has coordinates $x_1, \ldots, x_n$, then $\{\frac{\partial}{\partial x_i}\}$ are smooth sections of $TX \to X$ over U which define a trivialisation $TX|_U \simeq U \times \mathbb{R}^n$. Let $\{dx_i\}$ be the dual basis of sections of $T^*X \to X$ over U. Then $\omega|_U = \sum_{i_1 < \ldots < i_k} a_{i_1 \ldots i_k} dx_{i_1} \ldots dx_{i_k}$ where $a_{i_1 \ldots i_k} \in C^\infty(U)$.)

12.2 Kähler differentials

Definition 12.2. Let A be a finitely generated k-algebra. The *module of differentials of A/k* is the A-module $\Omega_{A/k}$ generated by formal symbols df for each $f \in A$ modulo the relations $d(fg) = gdf + fdg$, $d(f+g) = df + dg$, and $d\lambda = 0$ for $\lambda \in k$.

Equivalently, $\Omega_{A/k}$ can be defined using a universal property as follows. Let M be an A-module. A *derivation* of A/k into M is a map $D\colon A \to M$ such that $D(fg) = gDf + fDg$, $D(f+g) = Df + Dg$, and $D\lambda = 0$ for $\lambda \in k$. Thus $d\colon A \to \Omega_{A/k}$ is a derivation by construction, and satisfies the following universal property: if $D\colon A \to M$ is a derivation, there exists a unique A-module homomorphism $\theta\colon \Omega_{A/k} \to M$ such that $D = \theta \circ d$. Let

$\mathrm{Der}_k(A, M)$ denote the A-module of derivations of A/k into M. Then the universal property gives an identification

$$\mathrm{Der}_k(A, M) = \mathrm{Hom}_A(\Omega_{A/k}, M).$$

Example 12.3. If $A = k[x_1, \ldots, x_n]$ then

$$\Omega_{A/k} = A dx_1 \oplus \cdots \oplus A dx_n.$$

Proof. For $f \in A$ we have $df = \sum_{i=1}^n \frac{\partial f}{\partial x_i} dx_i$, so the dx_i generate. Suppose $\sum_{i=1}^n a_i dx_i = 0$. The map $D_i = \frac{\partial}{\partial x_i} \colon A \to A$ is a derivation such that $x_i \mapsto \delta_{ij}$. So, if $D_i = \theta_i \circ d$, then $0 = \theta_i(\sum a_j dx_j) = \sum a_j D_i x_j = a_i$. Thus $a_i = 0$ for all i. $\square$

Example 12.4. Let X be a variety over k of dimension n and $K = k(X)$. So K is an algebraic extension of the field $k(t_1, \ldots, t_n)$ of rational functions over k in n indeterminates $t_1, \ldots, t_n$. If $\mathrm{char}\, k = 0$ then

$$\Omega_{K/k} = K dt_1 \oplus \cdots \oplus K dt_n.$$

(If $\mathrm{char}\, k = p$, the same is true if the field extension $k(t_1, \ldots, t_n) \subset K$ is separable, and this holds for some choice of the t_i.)

Proof. We show that $dt_1, \ldots, dt_n$ span $\Omega_{K/k}$. First, if $f \in k(t_1, \ldots, t_n)$ then $df = \sum \frac{\partial f}{\partial t_i} dt_i$. Second, if $f \in K$, then f satisfies a polynomial

$$f^n + a_{n-1} f + \ldots + a_0 = 0$$

with $a_i \in k(t_1, \ldots, t_n)$. Applying d to this equation we deduce that $df \in \langle dt_1, \ldots, dt_n \rangle_K$. (We give the proof of linear independence in Rem. 12.8.) $\square$

Lemma 12.5. *Let A/k be a k-algebra and $S \subset A$ a multiplicative system. Then the natural map $S^{-1}\Omega_{A/k} \to \Omega_{S^{-1}A/k}$ is an isomorphism.*

Proof. Exercise. $\square$

Let X be a variety and $P \in X$ a point. We write $\Omega_{X,P} := \Omega_{\mathcal{O}_{X,P}/k}$.

Lemma 12.6. *There is a natural identification*

$$\Omega_{X,P} \otimes_{\mathcal{O}_{X,P}} k \xrightarrow{\sim} m_{X,P}/m_{X,P}^2 \quad df \mapsto \overline{(f - f(P))}$$

Proof. Recall $\Omega_{X,P}$ is an $\mathcal{O}_{X,P}$-module and $m_{X,P} \subset \mathcal{O}_{X,P}$ denotes the maximal ideal. In the statement the structure of $\mathcal{O}_{X,P}$-module on k is given by $k = \mathcal{O}_{X,P}/m_{X,P}$, the residue field of $\mathcal{O}_{X,P}$. So $\Omega_{X,P} \otimes k = \Omega_{X,P}/m_{X,P}\Omega_{X,P}$.

We have

$$\operatorname{Hom}_k(\Omega_{X,P} \otimes k, k) = \operatorname{Hom}_{\mathcal{O}_{X,P}}(\Omega_{X,P}, k) = \operatorname{Der}_k(\mathcal{O}_{X,P}, k)$$

where the first equality is the standard adjunction

$$\operatorname{Hom}_B(M \otimes_A B, N) = \operatorname{Hom}_A(M, N)$$

and the second is given by the universal property of Ω. We identify the k-vector space $\operatorname{Der}_k(\mathcal{O}_{X,P}, k)$ with $\operatorname{Hom}_k(m_{X,P}/m_{X,P}^2, k)$ as follows. Given $D\colon \mathcal{O}_{X,P} \to k$ a derivation, we have $D(m_{X,P}^2) = 0$ by the Leibniz rule and $D(k) = 0$. So D corresponds to a k-linear map $\theta\colon m_{X,P}/m_{X,P}^2 \to k$ given by $D(f) = \theta(\overline{f - f(P)})$. Conversely, given any θ, the map D defined in this way is a derivation. Hence $\operatorname{Hom}_k(\Omega_{X,P} \otimes k, k) = \operatorname{Hom}_k(m_{X,P}/m_{X,P}^2, k)$ and so $\Omega_{X,P} \otimes k = m_{X,P}/m_{X,P}^2$. $\square$

Proposition 12.7. *Let X be a variety and $P \in X$ a smooth point. Then $\Omega_{X,P}$ is a free $\mathcal{O}_{X,P}$-module of rank $n = \dim X$. If $t_1, \ldots, t_n$ is a system of local parameters at $P \in X$ (that is, $m_{X,P} = (t_1, \ldots, t_n)$) then $dt_1, \ldots, dt_n$ is a basis of $\Omega_{X,P}$.*

Proof. Since smoothness is an open property, we can replace X by a smooth open affine neighbourhood U of $P \in X$. So $m_{U,Q}/m_{U,Q}^2$ has dimension $n = \dim U$ for every $Q \in U$. Thus $\Omega_{U,Q} \otimes k$ has dimension n for all Q by 12.6. Pick a basis for $\Omega_{U,P} \otimes k$ and lift to elements of $\Omega_{U,P}$. These generate $\Omega_{U,P}$ as an $\mathcal{O}_{U,P}$-module by Nakayama's lemma, that is, define a surjection $\mathcal{O}_{U,P}^{\oplus n} \to \Omega_{U,P}$. Let $A = k[U]$. Then $\Omega_{U,P} = (\Omega_{A/k})_{m_P}$, the localisation of $\Omega_{A/k}$ at the corresponding maximal ideal $m_P \subset A$. So after a localisation $B = A_f$ for some $f \in A$, $f(P) \neq 0$ (corresponding to replacing U by a smaller open affine neighbourhood V of P, $V = (f \neq 0) \subset U$), we obtain a surjection $B^{\oplus n} \to \Omega_{B/k}$. Pick generators of the kernel to obtain an exact sequence

$$B^{\oplus m} \xrightarrow{\theta} B^{\oplus n} \to \Omega_B \to 0.$$

The map θ is given by a matrix $\theta = (\theta_{ij})$ of elements of B. Now tensor this exact sequence with the residue field $k(Q) = B/m_Q$ at a point $Q \in V$ to obtain the exact sequence

$$k^{\oplus m} \to k^{\oplus n} \to \Omega_{V,Q} \otimes k \to 0.$$

(Recall that the functor $\otimes_A M$ is right exact for all A, M). The k-vector space $\Omega_{V,Q} \otimes k$ has dimension n, so the second arrow is an isomorphism. Hence the first arrow is zero, that is, $\theta(Q) = (\theta_{ij}(Q)) = 0$. So $\theta = 0$ and the map $B^{\oplus n} \to \Omega_{B/k}$ is an isomorphism. Thus the map $\mathcal{O}_{X,P}^{\oplus n} \to \Omega_{X,P}$ given by localisation at P is also an isomorphism. $\square$

Remark 12.8. It follows that $\Omega_{k(X)/k} = k(X)dt_1 \oplus \cdots \oplus k(X)dt_n$. In particular, $\dim_{k(X)} \Omega_{k(X)/k} = n$. This completes the proof of 12.4.

Now let X be a smooth variety over k. Let $K = k(X)$. The K-vector space $\Omega_{K/k}$ is the space of *rational* 1*-forms* on X. Elements of $\Omega_{K/k}$ are finite sums $\sum f_i dg_i$ where f_i, g_i are rational functions on X (modulo the usual relations). For $P \in X$ the $\mathcal{O}_{X,P}$-module $\Omega_{X,P} := \Omega_{\mathcal{O}_{X,P}/k}$ is the module of rational 1-forms ω which are regular in a neighbourhood of $P \in X$, that is, $\omega = \sum f_i dg_i$ where the f_i, g_i are regular at P. (Note that the natural map $\Omega_{X,P} \to \Omega_{K/k}$ is injective because $\Omega_{K/k}$ is a localisation of $\Omega_{X,P}$ and $\Omega_{X,P}$ is a torsion-free $\mathcal{O}_{X,P}$-module by 12.7. This fails in general if $P \in X$ is not smooth.) For $U \subset X$ an open subset the regular 1-forms on U are

$$\Omega_X(U) := \bigcap_{P \in X} \Omega_{X,P} \subset \Omega_{K/k}$$

Lemma 12.9. *If U is an affine variety then $\Omega_U(U) = \Omega_{k[U]/k}$*

Proof. This is similar to the proof that the k-algebra of regular functions $\mathcal{O}_U(U)$ on an affine variety U equals the coordinate ring $k[U]$. Write $A = k[U]$. We certainly have an inclusion

$$\Omega_{A/k} \subseteq \Omega_U(U) = \bigcap_{P \in U} \Omega_{U,P}.$$

Suppose $\omega \in \Omega_U(U)$. Then, for each $P \in U$ we have $\omega = \sum f_{i,P} dg_{i,P}$, where $f_{i,P}, g_{i,P} \in \mathcal{O}_{U,P}$. Clearing denominators we obtain $a_P \omega = \sum b_{i,P} dc_{i,P}$, where $a_P, b_{i,P}, c_{i,P} \in A$ and $a_P(P) \neq 0$. Now, by the Nullstellensatz there exists a finite sum $\sum \rho_P a_P = 1$, some $\rho_P \in A$. Hence $\omega = \sum \rho_P b_{i,P} dc_{iP}$. $\square$

Example 12.10. Let $X = \mathbb{P}^1$. Let X_0, X_1 be the homogeneous coordinates, $U_0 = (X_0 \neq 0) = \mathbb{A}^1_x$, and $U_1 = (X_1 \neq 0) = \mathbb{A}^1_z$, where $x = X_1/X_0$ and $z = X_0/X_1 = x^{-1}$. Then $\Omega_{k(X)/k} = k(x)dx$, $\Omega_X(U_0) = \Omega_{k[x]/k} = k[x]dx$, and $\Omega_X(U_1) = k[z]dz = k[x^{-1}]x^{-2}dx$, so

$$\Omega_X(X) = \Omega_X(U_0) \cap \Omega_X(U_1) = 0.$$

Lemma 12.11. *Let A be a finitely generated k-algebra. Write $A = \frac{k[x_1,\dots,x_N]}{(f_1,\dots,f_r)}$. Then*

$$\Omega_{A/k} = \frac{Adx_1 \oplus \cdots \oplus Adx_N}{\langle df_1, \dots, df_r\rangle}.$$

Proof. Write $P = k[x_1,\dots,x_N]$ and $I = (f_1,\dots,f_r) \subset P$, so $A = P/I$. Let M be an A-module. Then

$$\begin{aligned}\operatorname{Hom}(\Omega_{A/k}, M) &= \operatorname{Der}_k(A,M) = \{D \in \operatorname{Der}_k(P,M) \mid D(I) = 0\} \\ &= \{\theta \in \operatorname{Hom}_P(\Omega_{P/k}, M) \mid \theta(dI) = 0\} \\ &= \operatorname{Hom}_P\left(\frac{\Omega_{P/k}}{\langle dI\rangle}, M\right) = \operatorname{Hom}_A\left(\frac{\Omega_{P/k}\otimes A}{\langle dI\rangle}, M\right).\end{aligned}$$

So

$$\Omega_{A/k} = \frac{\Omega_{P/k}\otimes A}{\langle dI\rangle} = \frac{Adx_1 \oplus \cdots \oplus Adx_n}{\langle df_1,\dots,df_r\rangle}.$$

□

Example 12.12. Let $X \to \mathbb{P}^1$ be a hyperelliptic curve. Then X is the union of two affine pieces $U_0 = (y^2 = f(x)) \subset \mathbb{A}^2_{x,y}$ and $U_1 = (w^2 = g(z)) \subset \mathbb{A}^2_{z,w}$ where $f(x) = \prod_{i=1}^{2g+2}(x-\alpha_i)$, $g(z) = z^{2g+2}f(z^{-1}) = \prod_{i=1}^{2g+2}(1-\alpha_i z)$ and the glueing is given by

$$U_0 \supset (x \neq 0) \xrightarrow{\sim} (z \neq 0) \subset U_1, \quad (x,y) \mapsto (z,w) = (x^{-1}, x^{-(g+1)}y).$$

(Here we chose the coordinate $x = X_1/X_0$ on $\mathbb{P}^1$ so that $X \to \mathbb{P}^1$ is not ramified over ∞). So, by 12.11,

$$\Omega_X(U_0) = \frac{k[U_0]dx \oplus k[U_0]dy}{\langle 2ydy - f'(x)dx\rangle}.$$

Observe that the rational 1-form $\frac{dx}{y}$ is regular on U_0. Indeed, $\frac{dx}{y} = \frac{2ydy}{f'(x)}$ and at any point $P \in U_0$ either $2y(P) \neq 0$ or $f'(P) \neq 0$ because U_0 is smooth (equivalently, $f(x)$ has no repeated roots). We deduce that $\Omega_X(U_0) = k[U_0]\frac{dx}{y}$. Similarly, $\Omega_X(U_1) = k[U_1]\frac{dz}{w}$. Now we compute $\Omega_X(X)$ using

$$\Omega_X(X) = \Omega_X(U_0) \cap \Omega_X(U_1) \subset \Omega_{k(X)/k}.$$

Note $\Omega_{k(X)/k} = k(X)\frac{dx}{y}$, $\frac{dz}{w} = \frac{d(x^{-1})}{x^{-(g+1)}y} = \frac{-x^{-2}dx}{x^{-(g+1)}y} = \frac{-x^{g-1}dx}{y}$, and $k(X) = k(x) \oplus k(x)y$. We find

$$\Omega_X(X) = k[U_0]\frac{dx}{y} \;\cap\; k[U_1]\frac{dz}{w} = k[x,y]\frac{dx}{y} \;\cap\; k[x^{-1}, x^{-(g+1)}y]\frac{x^{g-1}dx}{y}$$

$$= \langle 1, x, \dots, x^{g-1} \rangle_k \frac{dx}{y}.$$

Theorem 12.13. *Let X be smooth projective curve over $k = \mathbb{C}$. Then $\dim_k \Omega_X(X) = g$, the genus of X.*

We give the proof in Sec. 21. For a smooth projective curve over an arbitrary algebraically closed field k we *define* the genus by this formula.

12.3 Analytic viewpoint

Let X be a smooth complex variety, and X^{an} the associated complex manifold. Then for $U \subset X$ a Zariski open subset, the space $\Omega_X(U)$ of regular 1-forms on $U \subset X$ is a subspace of the space $\Omega_{X^{\mathrm{an}}}(U)$ of holomorphic 1-forms on $U \subset X^{\mathrm{an}}$ (that is, 1-forms on U locally (in the analytic topology) of the form $\sum f_i dz_i$ where $z_1, \dots, z_n$ are local complex coordinates and $f_i = f_i(z_1, \dots, z_n)$ are holomorphic). Moreover, if X is projective and $U = X$ we have equality: $\Omega_X(X) = \Omega_{X^{\mathrm{an}}}(X)$. See [Serre56].

13 1-forms on smooth curves

Let X be a smooth curve and $K = k(X)$. Let $P \in X$ be a point and t a local parameter at P. Then $\Omega_{X,P} = \mathcal{O}_{X,P} dt$ and $\Omega_{K/k} = K dt$. Let $\omega \in \Omega_{K/k}$ be a rational 1-form on X. Then $\omega = f dt$, some $f \in K$. We define $\nu_P(\omega)$, the *order of vanishing of ω at P*, to be the order of vanishing $\nu_P(f)$ of f at P. So

$$\omega = (a_\nu t^\nu + a_{\nu+1} t^{\nu+1} + \cdots) dt$$

where $\nu = \nu_P(\omega) \in \mathbb{Z}$. Note immediately that ω is regular at P iff $\nu_P(\omega) \geq 0$.

Theorem 13.1. *Let X be a smooth projective curve. Let ω be a rational 1-form on X. Then*

$$\sum_{P \in X} \nu_P(\omega) = 2g - 2$$

where g is the genus of X.

We give the proof in Sec. 21.

Example 13.2. Let $X \to \mathbb{P}^1$ be a hyperelliptic curve as in 12.12. Then the rational 1-form $\omega = \frac{dx}{y} = \frac{2dy}{f'(x)}$ has no zeroes or poles over the affine piece $\mathbb{A}^1_x = (X_0 \neq 0) \subset \mathbb{P}^1$ and has a zero of order $g - 1$ at each of the two points

P_1, P_2 over $\infty = (0 : 1) \in \mathbb{P}^1$. Indeed, rewriting ω in terms of $z = x^{-1}$ and $w = x^{-(g+1)}y$, we find

$$\omega = \frac{dx}{y} = \frac{d(z^{-1})}{z^{-(g+1)}w} = \frac{-z^{-2}dz}{z^{-(g+1)}w} = \frac{-z^{g-1}dz}{w}.$$

Now z is a local coordinate at P_i and $w(P_i) \neq 0$, so ω has a zero of order $(g-1)$ at P_i.

13.1 Residues

Definition 13.3. Let X be a smooth curve, $P \in X$ a point, and ω a rational 1-form on X. Let t be a local parameter at P and write

$$\omega = f dt = (a_\nu t^\nu + a_{\nu+1} t^{\nu+1} + \cdots)dt.$$

We define the *residue of* ω *at* P by $\operatorname{Res}_P \omega = a_{-1}$.

Lemma 13.4. $\operatorname{Res}_P \omega$ *is well defined (it does not depend on the choice of* t*).*

Proof 1 : Analysis ($k = \mathbb{C}$*).* Let γ be a small anticlockwise loop about $P \in X$, then

$$\operatorname{Res}_P \omega = \frac{1}{2\pi i} \int_\gamma \omega.$$

□

Proof 2 : Algebra. We assume that $\operatorname{char} k = 0$, see [Serre59, p. 20] for the case $\operatorname{char} k > 0$. Write $\operatorname{Res}_{P,t} \omega$ for the residue defined as above, where t is a local parameter at P. Then $\operatorname{Res}_{P,t}$ satisfies the following properties:

(1) $\operatorname{Res}_{P,t}$ is k-linear.

(2) $\operatorname{Res}_{P,t} \omega = 0$ if $\nu_P(\omega) \geq 0$.

(3) $\operatorname{Res}_{P,t}(df) = 0$ for $f \in K$

(1) and (2) are clear. To prove (3), write $f = b_\nu t^\nu + b_{\nu+1} t^{\nu+1} + \cdots$, then $df = \nu b_\nu t^{\nu-1} + \cdots$ and so the coefficient of $t^{-1}dt$ vanishes.

Now let s be another local parameter at $P \in X$. So $t = us$ where $u \in \mathcal{O}_{X,P}$ is a unit. We compute $\operatorname{Res}_{P,s} \omega$ in terms of the expansion $\omega = (a_\nu t^\nu + \cdots)dt$. By (1) and (2) we need only compute the residue of $t^i dt$ for each $i < 0$. Now $t^i dt = d(\frac{t^{i+1}}{i+1})$ for $i \neq -1$ (using $\operatorname{char} k = 0$) so $\operatorname{Res}_{P,s} t^i dt = 0$ by (3). Finally $t^{-1}dt = s^{-1}ds + u^{-1}du$, so $\operatorname{Res}_{P,s} t^{-1}dt = \operatorname{Res}_{P,s} s^{-1}ds = 1$ by (2). Hence $\operatorname{Res}_{P,s} \omega = a_{-1} = \operatorname{Res}_{P,t} \omega$ □

Theorem 13.5. *Let X be a smooth projective curve and ω a rational 1-form on X. Then $\sum_{P\in X} \operatorname{Res}_P \omega = 0$.*

Proof 1: Analysis ($k = \mathbb{C}$). For each pole $P_i \in X$ of ω let $D_i \subset X$ be a small disc centred at P_i and $\gamma_i = \partial D_i$ the small anticlockwise loop about P_i given by the boundary of D_i. Let $Y = X \setminus \bigcup D_i$. Then

$$\sum_{P\in X} \operatorname{Res}_P \omega = \frac{1}{2\pi i}\sum_i \int_{\gamma_i} \omega = \frac{-1}{2\pi i}\int_{\partial Y} \omega = \frac{-1}{2\pi i}\int_Y d\omega$$

by Stokes' theorem. Finally $d\omega = 0$ on Y because ω is holomorphic: locally $\omega = f dz$, where z is a complex coordinate and $f = f(z)$ is holomorphic, so

$$d\omega = df \wedge dz = \left(\frac{\partial f}{\partial z} dz + \frac{\partial f}{\partial \bar{z}} d\bar{z}\right) \wedge dz = \frac{\partial f}{\partial z} dz \wedge dz = 0$$

because $\frac{\partial f}{\partial \bar{z}} = 0$ (the Cauchy–Riemann equations). So $\sum_{P\in X} \operatorname{Res}_P \omega = 0$ as required. □

Proof 2: Algebra. We assume $\operatorname{char} k = 0$, see [Serre59, p. 15] for the case $\operatorname{char} k > 0$. We reduce to the case $X = \mathbb{P}^1$ and check explicitly in this case.

For $X = \mathbb{P}^1$, let $x = \frac{X_1}{X_0}$ be an affine coordinate and write $\omega = f(x)dx$ where $f(x) \in k(x)$. Use "partial fractions" to write

$$f(x) = \sum_{i\geq 0} a_i x^i + \sum_{i<0, \alpha\in k} \frac{b_{i,\alpha}}{(x-\alpha)^i}.$$

So, we may assume $\omega = x^n dx$ for some $n \in \mathbb{Z}$ by k-linearity of the residue (and since $\frac{1}{(x-\alpha)^i}dx$ corresponds to $x^{-i}dx$ under the change of coordinates $x \mapsto x - \alpha$). We enumerate the poles and compute the residues in each case. Let $z = x^{-1}$, a local coordinate at $\infty = (0:1) \in \mathbb{P}^1$. Then

$$\omega = x^n dx = -\frac{1}{z^{n+2}} dz.$$

So, if $n \geq 0$ then ω has a pole at ∞ with residue 0, if $n = -1$ then ω has poles at 0 and ∞ with residues 1 and -1, and if $n \leq -2$ then ω has a pole at 0 with residue 0. Thus in each case $\sum_{P\in X} \operatorname{Res}_P \omega = 0$.

Let $f\colon X \to Y$ be a morphism of smooth projective curves and assume that the corresponding field extension $k(Y) \subset k(X)$ is separable (this is automatic if $\operatorname{char} k = 0$). We define the *trace map*

$$\operatorname{Tr}\colon \Omega_{k(X)/k} \to \Omega_{k(Y)/k}$$

and show that, for ω a rational 1-form on X, we have

$$\sum_{P \in X} \operatorname{Res}_P \omega = \sum_{Q \in Y} \operatorname{Res}_Q (\operatorname{Tr} \omega).$$

Given a smooth projective curve X, there exists a separable extension $k(t) \subset k(X)$ corresponding to a map $X \to \mathbb{P}^1$, so we deduce our result from the case of $\mathbb{P}^1$.

We first define the trace map on functions $\operatorname{Tr}\colon k(X) \to k(Y)$. The field $k(X)$ is a $k(Y)$-vector space of dimension $d = \deg f$. Pick a basis. Multiplication by an element $g \in k(X)$ defines a $k(Y)$-linear map $k(X) \to k(X)$ given by a matrix $A(g) \in k(Y)^{d \times d}$ which is well defined up to conjugation by an element of $\mathrm{GL}_d(k(Y))$ (corresponding to change of basis). Thus the trace $\operatorname{Tr} A(g)$ of $A(g)$ is a well defined element of $k(Y)$, and we define

$$\operatorname{Tr}\colon k(X) \to k(Y), \quad g \mapsto \operatorname{Tr} A(g).$$

We note that the trace map Tr is $k(Y)$-linear and satisfies $\operatorname{Tr}|_{k(Y)} = d \cdot \mathrm{id}$, where $\mathrm{id}\colon k(Y) \to k(Y)$ is the identity map. Geometrically:

Lemma 13.6. *For $U \subset X$ an open subset we have $\operatorname{Tr}\colon \mathcal{O}_X(f^{-1}U) \to \mathcal{O}_Y(U)$, and*

$$\operatorname{Tr}(g)(Q) = \sum_{P \in f^{-1}Q} e_P g(P)$$

for $g \in \mathcal{O}_X(f^{-1}U)$ and $Q \in U$.

Proof of Lem. 13.6. For $Q \in Y$ a point, the ring $\bigcup_{P \in f^{-1}Q} \mathcal{O}_{X,P}$ is a free $\mathcal{O}_{Y,Q}$-module of rank $d = \deg f$. Hence we can define a trace map

$$\operatorname{Tr}_Q\colon \bigcup_{P \in f^{-1}Q} \mathcal{O}_{X,P} \to \mathcal{O}_{Y,Q}$$

as above, and the following diagram commutes.

$$\begin{array}{ccc} \bigcup_{P \in f^{-1}Q} \mathcal{O}_{X,P} & \xrightarrow{\operatorname{Tr}_Q} & \mathcal{O}_{Y,Q} \\ \cap & & \cap \\ k(X) & \xrightarrow{\operatorname{Tr}} & k(Y) \end{array}$$

In particular it follows that $\operatorname{Tr}(\mathcal{O}_X(f^{-1}U)) \subset \mathcal{O}_Y(U)$ for $U \subset Y$ open. To compute the trace map explicitly we pass to completions:

$$\begin{array}{ccc} \bigcup_{P \in f^{-1}Q} \mathcal{O}_{X,P} & \xrightarrow{\operatorname{Tr}_Q} & \mathcal{O}_{Y,Q} \\ \cap & & \cap \\ \bigoplus_{P \in f^{-1}Q} \hat{\mathcal{O}}_{X,P} & \xrightarrow{\hat{\operatorname{Tr}}_Q} & \hat{\mathcal{O}}_{Y,Q} \end{array}$$

Let $P \in f^{-1}Q$. We can pick local analytic coordinates $y \in \hat{\mathcal{O}}_{Y,Q}$ at $Q \in Y$ and $x \in \hat{\mathcal{O}}_{X,P}$ at $P \in X$ such that $y = x^e$ where e is the ramification index of f at P (using $\operatorname{char} k = 0$). Then

$$\hat{\mathcal{O}}_{Y,Q} = k[[y]] = k[[x^e]] \subset k[[x]] = \hat{\mathcal{O}}_{X,P}$$

Now we can compute the trace map $\hat{\operatorname{Tr}}_Q^P \colon \hat{\mathcal{O}}_{X,P} \to \hat{\mathcal{O}}_{Y,Q}$ explicitly: $k[[x]]$ has basis $1, x, \ldots, x^{e-1}$ as a $k[[y]]$-module, and multiplication by x has matrix

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & x^e \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}$$

with respect to this basis. We find $\hat{\operatorname{Tr}}_Q^P(x^i) = e$ if $i = 0$ and $\hat{\operatorname{Tr}}_Q^P(x^i) = 0$ if $0 < i < e$, so

$$\hat{\operatorname{Tr}}_Q^P\left(\sum_i c_i x^i\right) = e \sum_j c_{ej} x^{ej} = e \sum_j c_{ej} y^j.$$

Adding the contributions from each $P \in f^{-1}Q$ gives $\operatorname{Tr}(g)(Q) = \sum e_P g(P)$ as claimed. □

We next define the trace map on 1-forms. Given $k(Y) \subset k(X)$ a separable extension, write $k(Y)$ as a separable extension of $k(t)$. Then $\Omega_{k(Y)/k} = k(Y)dt$, $\Omega_{k(X)/k} = k(X)dt$, and we define $\operatorname{Tr} \colon \Omega_{k(X)/k} \to \Omega_{k(Y)/k}$ by $\operatorname{Tr}(g\,dt) = \operatorname{Tr}(g)dt$. This is independent of the choice of t because $\operatorname{Tr} \colon k(X) \to k(Y)$ is $k(Y)$-linear.

Finally, we show that $\sum_{P \in X} \operatorname{Res}_P(\omega) = \sum_{Q \in Y} \operatorname{Res}_Q(\operatorname{Tr}\omega)$. We reduce to a local computation for $(P \in X) \to (Q \in Y)$ given by $x \mapsto y = x^e$ as above. Write

$$\omega = \left(\sum_i a_i x^i\right) dx = \left(\frac{1}{e} \sum_i a_i x^{i-(e-1)}\right) dy$$

then

$$\hat{\operatorname{Tr}}_Q^P(\omega) = \hat{\operatorname{Tr}}_Q^P\left(\frac{1}{e}\sum_i a_i x^{i-(e-1)}\right) dy = \left(\sum_j a_{ej+(e-1)} y^j\right) dy$$

where $ej = i - (e-1)$. We deduce that $\operatorname{Res}_Q(\hat{\operatorname{Tr}}_Q^P \omega) = a_{-1} = \operatorname{Res}_P(\omega)$. The result follows. □

14 Morphisms from smooth curves to projective space

Proposition 14.1. *Let X be a smooth curve. A morphism $f\colon X \to \mathbb{P}^N$ is given by a $(N+1)$-tuple $(f_0, \ldots, f_N)$ of rational functions on X (not all zero). Two $(N+1)$-tuples $(f_0, \ldots, f_N)$ and $(g_0, \ldots, g_N)$ determine the same morphism iff $g_i = hf_i$ for some $h \in k(X)^\times$.*

Proof. The assignment

$$P \mapsto (f_0(P) : \cdots : f_N(P))$$

defines a morphism $U \to \mathbb{P}^N$ where $U \subset X$ is the open set of points $P \in X$ where all f_i are regular and some f_i is nonzero. This extends to a morphism $f\colon X \to \mathbb{P}^N$ because X is a smooth curve by Prop. 3.1 (and the extension is unique by the separatedness condition). Conversely if $f\colon X \to \mathbb{P}^N$ is a morphism, then X is not contained in the coordinate hyperplane $(X_i = 0)$ for some i, say $i = 0$. Then f is given by the rational functions $f_i = f^\sharp(X_i/X_0)$, the pullback of the rational functions X_i/X_0 on $\mathbb{P}^N$.

Recall that $\mathbb{P}^N = (k^{N+1} - \{0\})/k^\times$ (as a set) by definition. So, the two maps $f = (f_0 : \cdots : f_N)$ and $g = (g_0 : \cdots : g_n)$ are equal iff $g_i = hf_i$ on some open set $U \subset X$, where $h : U \to k^\times$. Then $h = g_i/f_i$ is a rational function on X. $\square$

The action of $\mathrm{GL}_{N+1}(k)$ on k^{N+1} induces an action of $\mathrm{PGL}_{N+1}(k) = \mathrm{GL}_{N+1}(k)/k^\times$ on $\mathbb{P}^N_k$. (In fact, this is the full automorphism group of $\mathbb{P}^N_k$.) Prop. 14.1 gives a correspondence between morphisms $f\colon X \to \mathbb{P}^N$ such that $f(X)$ is not contained in a hyperplane, modulo $\mathrm{PGL}_{N+1}(k)$ acting by composition, and k-vector spaces $V \subset k(X)$ of dimension $N+1$, modulo $k(X)^\times$ acting by multiplication, given by

$$f \mapsto V = \langle f_0, \ldots, f_N \rangle \subset k(X).$$

Indeed, $f(X)$ is not contained in a hyperplane iff $f_0, \ldots, f_N$ are linearly independent over k, and changing the basis $f_0, \ldots, f_N$ of V corresponds to composing f with some $\theta \in \mathrm{PGL}_n(k)$. To better understand this correspondence we introduce the notion of a "linear system".

15 Divisors and the class group

Let X be a smooth curve. A *divisor* D on X is a finite formal sum of points of X, $D = \sum_{P \in X} n_P P$, $n_P \in \mathbb{Z}$. The *degree* of D is $\deg D := \sum_{P \in X} n_P$. We say D is *effective* and write $D \geq 0$ if $n_P \geq 0$ for all P.

If $f \in k(X)^\times$ is a non-zero rational function on X, the *principal divisor* (f) is defined by

$$(f) := \sum_{P \in X} \nu_P(f) P.$$

Thus (f) is the sum of the zeroes of f minus the sum of the poles of f, with coefficients given by the multiplicities. Note that $(fg) = (f) + (g)$.

Lemma 15.1. *Assume X is projective. Then* $\deg(f) = 0$.

Proof. f defines a morphism $F = (1 : f) \colon X \to \mathbb{P}^1$. The sets of zeroes and poles of f are $F^{-1}(1 : 0)$ and $F^{-1}(0 : 1)$. These sets both have size $d = \deg F$ if we count with multiplicities by Prop. 3.6. So $\deg(f) = 0$. $\square$

We say two divisors D and D' are *linearly equivalent* and write $D \sim D'$ if $D' - D$ is principal, that is, $D' = D + (f)$ for some f. The *(divisor) class group* $\mathrm{Cl}(X)$ of X is the quotient of the group $\mathrm{Div}(X)$ of divisors on X by linear equivalence. By the lemma, if X is projective then the homomorphism $\deg \colon \mathrm{Div}(X) \to \mathbb{Z}$ descends to a homomorphism $\deg \colon \mathrm{Cl}(X) \to \mathbb{Z}$.

Remark 15.2. In algebraic number theory, if K is a number field (a finite extension of $\mathbb{Q}$) and $\mathcal{O} \subset K$ is the ring of integers (the integral closure of $\mathbb{Z}$ in K), then the *(ideal) class group* $\mathrm{Cl}(\mathcal{O})$ of $\mathcal{O}$ is defined as the quotient of the multiplicative group of nonzero fractional ideals $I \subset K$ by the subgroup of principal ideals. Here a *fractional ideal* I is a finitely generated $\mathcal{O}$-submodule of K, and we say I is *prinicpal* if $I = \mathcal{O} \cdot f$ for some $f \in K$.

Now suppose K is the function field of a curve X over an algebraically closed field k. Let $U \subset X$ be an affine piece and $A = k[U]$ the coordinate ring of U. Then one can define the ideal class group $\mathrm{Cl}(A)$ as above, and $\mathrm{Cl}(A)$ coincides with the divisor class group $\mathrm{Cl}(U)$.

Example 15.3. Let $X = \mathbb{P}^1$. Then $\deg \colon \mathrm{Cl}(\mathbb{P}^1) \to \mathbb{Z}$ is an isomorphism. Indeed, it suffices to show that if $P, Q \in X$ then $P \sim Q$. Let $f = \frac{x - x(P)}{x - x(Q)}$ where x is an affine coordinate on $\mathbb{P}^1$, then $(f) = P - Q$.

Here we briefly describe (without proofs) the structure of $\mathrm{Cl}(X)$ for a smooth projective curve over $k = \mathbb{C}$ in general. Let $\mathrm{Cl}^0(X)$ be the kernel of $\deg \colon \mathrm{Cl}(X) \longrightarrow \mathbb{Z}$, that is, the group of linear equivalence classes of divisors of degree 0 on X. Then there is a natural isomorphism

$$A \colon \mathrm{Cl}^0(X) \xrightarrow{\sim} \frac{\Omega_X(X)^*}{H_1(X, \mathbb{Z})} =: J(X)$$

$J(X)$ is the *Jacobian* of X and A is the *Abel-Jacobi* map.

First we describe the Jacobian $J(X)$ (a compact complex Lie group). Recall that $\Omega_X(X)$ is a $\mathbb{C}$-vector space of dimension g where g is the genus of X. Also $H_1(X,\mathbb{Z})$, the first homology group of X, is a free abelian group of rank $2g$. The map $H_1(X,\mathbb{Z}) \to \Omega_X(X)^*$ is given by integration: recall that an element of $H_1(X,\mathbb{Z})$ is a 1-cycle (a linear combination of closed loops on X) modulo boundaries of 2-dimensional regions $D \subset X$. So, by Stokes' theorem, integration of a 1-form ω such that $d\omega = 0$ over a 1-cycle γ only depends on the class of γ in $H_1(X,\mathbb{Z})$ (if γ is the boundary of a region D then $\int_\gamma \omega = \int_D d\omega = 0$). Recall that a holomorphic form on a Riemann surface satisfies $d\omega = 0$. So we obtain a map

$$H_1(X,\mathbb{Z}) \to \Omega_X(X)^*, \quad \gamma \mapsto \int_\gamma \omega.$$

The induced map

$$H_1(X,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R} \to \Omega_X(X)^*$$

is an isomorphism of $\mathbb{R}$-vector spaces of dimension $2g$. (The proof of this fact uses Hodge theory and is outside the scope of this course, see [GH, p. 116, p. 227-8]). It follows that the quotient $J(X) = \Omega_X(X)^*/H_1(X,\mathbb{Z})$ is isomorphic as a smooth manifold to the real $2g$-torus $(\mathbb{R}/\mathbb{Z})^{2g} = (S^1)^{2g}$. As a complex manifold, $J(X)$ is a *complex torus* $\mathbb{C}^g/L$, where $L \subset \mathbb{C}^g$, $L \simeq \mathbb{Z}^{2g}$, and its isomorphism type depends on L.

Next we describe the map A. First, fix a basepoint $P_0 \in X$. We define

$$A \colon X \to J(X), \quad P \mapsto \left(\omega \mapsto \int_{P_0}^P \omega\right).$$

That is, given a point $P \in X$, we define an element $\theta \in \Omega_X(X)^*$ as follows: choose a path α from P_0 to P on X and define $\theta(\omega) = \int_\alpha \omega$. The functional θ depends on the choice of α: if α' is another path from P_0 to P, then $\theta'(\omega) = \theta(\omega) + \int_\gamma \omega$, where γ is the loop $\alpha' \cdot \alpha^{-1}$. Thus $\theta = \theta'$ in $J(X) = \Omega_X(X)^*/H_1(X,\mathbb{Z})$, so the assignment $P \mapsto \theta$ gives a well-defined map $A \colon X \to J(X)$. To make things more explicit, pick a basis $\omega_1, \ldots, \omega_g$ of $\Omega_X(X)$. Then, with respect to the dual basis of $\Omega_X(X)$, the Jacobian $J(X)$ is the complex torus $\mathbb{C}^g/L$ where

$$L = \left\{ \left(\int_\gamma \omega_1, \ldots, \int_\gamma \omega_g\right) \;\middle|\; \gamma \in H_1(X,\mathbb{Z}) \right\} \subset \mathbb{C}^g$$

and the Abel-Jacobi map is

$$A \colon X \to \mathbb{C}^g/L, \quad P \mapsto \left(\int_{P_0}^P \omega_1, \ldots, \int_{P_0}^P \omega_g\right).$$

Now we extend $A\colon X \to J(X)$ to a map $A\colon \mathrm{Div}(X) \to J(X)$ by linearity, that is, $A(\sum n_P P) := \sum n_P A(P)$. Restricting to divisors of degree 0 we obtain a map $A\colon \mathrm{Div}^0(X) \to J(X)$ which is canonically determined (it does not depend on the choice of a base point). Indeed, suppose we change the basepoint $P_0 \in X$ to P_0'. Let α be a path from P_0 to P_0' on X. Then, in coordinates as above

$$A'(P) = A(P) + \left(\int_\alpha \omega_1, \ldots, \int_\alpha \omega_g \right),$$

that is, A and A' differ by a translation. So $A'(\sum n_i P_i) = A(\sum n_i P_i)$ if $\sum n_i = 0$. We state two theorems without proof (see [GH, p. 235]).

Theorem 15.4 (Abel's theorem). *For $D \in \mathrm{Div}^0(X)$, $A(D) = 0$ iff $D \sim 0$.*

Theorem 15.5 (Jacobi inversion theorem). $A\colon \mathrm{Div}^0(X) \to J(X)$ *is surjective.*

Together, 15.4 and 15.5 show that $A\colon \mathrm{Div}^0(X) \to J(X)$ descends to an isomorphism $A\colon \mathrm{Cl}^0(X) \xrightarrow{\sim} J(X)$ as stated above.

16 Linear systems

Let X be a smooth projective curve and $D = \sum n_P P$ a divisor on X. The *complete linear system associated to D* is the set

$$|D| := \{D' \sim D \mid D' \geq 0\}$$

of effective divisors D' linearly equivalent to D. We also define the k-vector space

$$L(D) := \{f \in k(X) \mid f = 0 \text{ or } D + (f) \geq 0\} \subset k(X).$$

The condition $D + (f) \geq 0$ says $\nu_P(f) \geq -n_P$ for all $P \in X$. So, if D is effective (for example), $L(D)$ consists of rational functions having at pole of order at worst n_P at P for each $P \in X$. Observe that

$$\mathbb{P}L(D) := (L(D) - \{0\})/k^\times \xrightarrow{\sim} |D|, \quad f \mapsto D + (f).$$

(Proof: If $(f) = (g)$ then $(f/g) = 0$, so $f/g \in k^\times$ because $\mathcal{O}_X(X) = k$). In particular $|D|$ is a projective space over k. We write $l(D) := \dim_k L(D)$.

Example 16.1. Let $X = \mathbb{P}^1$ and $D = nP$, where $P = (0:1)$. Let $x = X_1/X_0$ be the affine coordinate on $X \setminus P$. Then $L(D) = \langle 1, x, \ldots, x^n \rangle_k$.

Example 16.2. Let $X = E$ be an elliptic curve (a curve of genus 1). Then the associated complex manifold E^{an} is a complex torus $\mathbb{C}/L$. Here $L \subset \mathbb{C}$ is a lattice (that is, $L \simeq \mathbb{Z}^2$ as an abelian group, and L generates $\mathbb{C}$ as an $\mathbb{R}$-vector space). Let $P \in E$ be the origin $0 \in \mathbb{C}/L$. Then $L(P) = k$ by Lem. 16.3(4). We construct an nonconstant rational function $\wp \in L(2P)$, the *Weierstrass $\wp$-function*. The field of meromorphic functions on the complex manifold E^{an} coincides with the field of rational functions on E (cf. [Serre56]). Let z be the coordinate on the cover $\mathbb{C} \to \mathbb{C}/L$. We define

$$\wp(z) = \sum_{l \in L \setminus \{0\}} \left(\frac{1}{(z-l)^2} - \frac{1}{l^2} \right) + \frac{1}{z^2}.$$

Then the sum converges absolutely for $z \in \mathbb{C} \setminus L$ and defines a meromorphic function $\wp \colon \mathbb{C} \dashrightarrow \mathbb{C}$ which has a pole of order 2 at each $l \in L$ and is regular elsewhere. (Warning: the sums $\sum \frac{1}{(z-l)^2}$ and $\sum \frac{1}{l^2}$ do not converge. So one needs to be careful when manipulating this expression). Also, $\wp(z+l) = \wp(z)$ for $l \in L$, so $\wp$ descends to a meromorphic function on $E^{\mathrm{an}} = \mathbb{C}/L$ with a single pole of order 2 at $P \in E$. Thus $\wp \in L(2P)$ as required. We deduce $L(2P) = \langle 1, \wp \rangle_k$ by Lem. 16.3(3). The function $\wp$ defines a degree 2 morphism

$$f = (1 : \wp) \colon E \to \mathbb{P}^1$$

with $f^{-1}(0:1) = 2P$. (So, the morphism f is branched over $(0:1)$ and 3 other points by the Riemann–Hurwitz formula.)

Now consider $L(3P)$. We observe that the derivative $\wp'$ of $\wp$ with respect to z lies in $L(3P)$. So $L(3P) = \langle 1, \wp, \wp' \rangle_k$ by 16.3(3). We obtain a morphism $g = (1 : \wp : \wp') : E \to \mathbb{P}^2$, which embeds E as a plane curve of degree 3, such that $(X_0 = 0) \cdot E = 3P$, that is, the line $(X_0 = 0) \subset \mathbb{P}^2$ is tangent to E at P with contact order 3 (so $P \in E$ is an inflection point in this embedding).

For $D = \sum_{P \in X} n_P P$ an effective divisor, the *support* $\mathrm{Supp}(D)$ of D is the set of points $P \in X$ such that $n_P > 0$.

Lemma 16.3. *(1) If $D \sim D'$ then $L(D) \xrightarrow{\sim} L(D')$.*

(2) If $\deg D < 0$ then $L(D) = 0$.

(3) If $\deg D = 0$ then $l(D) = 1$ if $D \sim 0$ and $l(D) = 0$ otherwise.

(4) If $\deg D > 0$ then $l(D) \le \deg D + 1$ with equality iff $X \simeq \mathbb{P}^1$.

Proof. If $D \sim D'$ write $D' = D + (f)$, then

$$L(D) \xrightarrow{\sim} L(D'), \quad g \mapsto f^{-1} g$$

because $D' + (f^{-1}g) = D + (g)$.

If $\deg D < 0$ and $f \in k(X)^\times$ then $\deg(D + (f)) = \deg D < 0$, so $D + (f)$ cannot be effective. So $L(D) = 0$. If $\deg D = 0$ then $\deg(D + (f)) = 0$, so $D + (f) \geq 0$ iff $D + (f) = 0$, that is, $D = (f^{-1})$.

To prove (4), we may assume D is effective (because $l(D)$ only depends on the linear equivalence class of D and if D is not linearly equivalent to an effective divisor then $L(D) = 0$). Let P be a point in the support of D. Then we have an exact sequence

$$0 \to L(D - P) \to L(D) \xrightarrow{\theta} k$$

where the last arrow θ is defined as follows: fix z a local coordinate at P. Given $f \in L(D)$ write

$$f = a_{-n_P} z^{-n_P} + a_{-n_P+1} z^{-n_P+1} + \cdots ,$$

and define $\theta(f) = a_{-n_P}$. Thus $l(D) \leq l(D - P) + 1$, and we deduce $l(D) \leq \deg D + 1$ by induction (note $L(0) = k$). Finally if we have equality $l(D) = \deg D + 1$, then $l(P) = 2$ for some P. Write $L(D) = \langle 1, f \rangle_k$, then the morphism $F = (1 : f)\colon X \to \mathbb{P}^1$ has degree 1 (because $F^{-1}(0 : 1) = P$ with multiplicity 1), that is, F is an isomorphism. $\square$

The Riemann–Roch theorem (Thm. 21.2) determines $l(D) = \dim_k L(D)$. For example, it follows from the Riemann–Roch theorem that

$$l(D) = 1 - g + \deg D \quad \text{ if } \deg D \geq 2g - 1$$

where g is the genus of X.

A *linear system* $\mathfrak{d}$ is a projective linear subspace of a complete linear system. We say $\mathfrak{d}$ is *basepoint free* if for all $P \in X$ there exists $D \in \mathfrak{d}$ such that $P \notin \operatorname{Supp} D$.

Example 16.4. Let $X \subset \mathbb{P}^N$ be a smooth projective curve embedded in projective space. Assume X is not contained in a hyperplane. For $H \subset \mathbb{P}^N$ a hyperplane, we define a divisor $D = X \cdot H$ as follows: locally on $\mathbb{P}^N$, say on $U \subset \mathbb{P}^N$, we have $H = (g = 0)$, where g is a regular function on U, and we define $D|_U = (g)$. Thus D is an effective divisor with support the set $X \cap H$, and the coefficient n_P is the "order of contact" of H with X at P. In, particular, if H meets X transversely, then $n_P = 1$ for all $P \in X \cap H$. Now let

$$\mathfrak{d} = \{X \cdot H \mid H \subset \mathbb{P}^N \text{ a hyperplane}\}.$$

Then $\mathfrak{d}$ is a basepoint free linear system on X. Indeed, if H and H' are two hyperplanes, then $H = (L = 0)$, $H' = (L' = 0)$ for L, L' linear forms on $\mathbb{P}^N$, and $X \cdot H - X \cdot H' = (h)$ where $h \in k(X)$ is the restriction of the rational function L/L' on $\mathbb{P}^N$. So elements of $\mathfrak{d}$ are linearly equivalent effective divisors. Finally it's clear that $\mathfrak{d}$ is basepoint free because for any $P \in X$ there exists a hyperplane $H \subset \mathbb{P}^N$ not containing P.

Proposition 16.5. *Let X be a smooth projective curve. There is a bijection between morphisms $f\colon X \to \mathbb{P}^N$ such that $f(X)$ is not contained in a hyperplane, modulo PGL_{N+1}, and basepoint free linear systems $\mathfrak{d}$ on X of dimension N.*

Proof. First we describe the correspondence. Given f, we define

$$\mathfrak{d} = \{f^*H \mid H \subset \mathbb{P}^N \text{ a hyperplane }\}.$$

Here the pullback f^*H of H to X is defined as follows (cf. 16.4). Locally on $\mathbb{P}^N$, say on $U \subset \mathbb{P}^N$, we have $H = (g = 0)$, where g is a regular function on U. We define f^*H on $f^{-1}U$ by $f^*H|_{f^{-1}U} = (f^\sharp(g))$ (where $f^\sharp(g) := g \circ f$). Thus f^*H has support $f^{-1}H$ and coefficients given by the above formula, that is, $n_P = \nu_P(f^\sharp(g))$ for $P \in U$. If $H, H' \subset \mathbb{P}^N$ are two hyperplanes, write $H = (L = 0)$, $H' = (L' = 0)$, where L, L' are linear forms on $\mathbb{P}^N$, then $f^*H' = f^*H + (h)$ where $h = f^\sharp(L'/L)$.

Conversely, given $\mathfrak{d}$, pick $D \in \mathfrak{d}$. Then $\mathfrak{d} \subseteq |D|$ corresponds to $V \subseteq L(D)$, a k-vector subspace. Pick a basis $f_0, \dots, f_N$ of V, and let f be the morphism $f = (f_0 : \cdots : f_N)$.

Now we check the two constructions are inverse. First suppose $\mathfrak{d}$ is given and let f be the map defined as above, we must show that the divisors f^*H are exactly the elements of $\mathfrak{d}$. Let $H = (\sum \alpha_i X_i = 0)$, where the X_i are the homogeneous coordinates on $\mathbb{P}^N$. We claim $f^*H = D + (\sum \alpha_i f_i) \in \mathfrak{d}$. Indeed, for $P \in X$, choose j such that $\nu_P(f_j)$ is minimal. Then, near P, f_i/f_j is regular for each i, and $f = (f_0/f_j : \cdots : f_N/f_j)$. So $f(P) \in (X_j \neq 0) \subset \mathbb{P}^N$, and the hyperplane H is defined near $f(P)$ by $H = (\sum \alpha_i (X_i/X_j) = 0)$. So, near P, $f^*H = (f^\sharp(\sum \alpha_i(X_i/X_j))) = (\sum \alpha_i(f_i/f_j)) = (\sum \alpha_i f_i) - (f_j)$. Finally, $\mathfrak{d}$ is basepoint free and $\nu_P(f_j)$ is minimal, so $P \notin \operatorname{Supp}(D + (f_j))$, that is, $D + (f_j) = 0$ near P. Thus $f^*H = D + (\sum \alpha_i f_i)$ near P, and so $f^*H = D + (\sum \alpha_i f_i)$ as claimed.

Finally suppose f is given and $\mathfrak{d}$ is the linear system given by pullbacks of hyperplanes. Let $D = f^*(X_0 = 0) \in \mathfrak{d}$, then $\mathfrak{d} = \{D + (\sum \alpha_i g_i)\}$ where $g_i = f^\sharp(X_i/X_0) = f_i/f_0$. Then $(g_0 : \cdots : g_N) = (f_0 : \cdots : f_N) = f$. $\square$

Remark 16.6. We usually study complete linear systems for the following reason. If $\mathfrak{d} \subsetneq |D|$, let $\phi_{\mathfrak{d}} \colon X \to \mathbb{P}^N$, $\phi_{|D|} \colon X \to \mathbb{P}^M$ be the corresponding morphisms. Here $N = \dim \mathfrak{d} < M = \dim |D|$. Then $\phi_{\mathfrak{d}} = p \circ \phi_{|D|}$ where $p \colon \mathbb{P}^M \dashrightarrow \mathbb{P}^N$ is a projection and p is a morphism near $\phi_{|D|}(X)$. That is, in suitable coordinates,

$$p \colon \mathbb{P}^M \dashrightarrow \mathbb{P}^N, \quad (X_0 : \ldots : X_M) \mapsto (X_0 : \ldots : X_N)$$

and $\phi_{|D|}(X)$ is disjoint from the locus $(X_0 = \cdots = X_N = 0) \subset \mathbb{P}^M$ where p is not defined.

Remark 16.7. Suppose $\mathfrak{d}$ is a linear system on a curve X which is not basepoint free. Let F be the *fixed part* of $\mathfrak{d}$, that is, the largest effective divisor F such that $D \geq F$ for all $D \in \mathfrak{d}$. Then

$$\mathfrak{d}' = \mathfrak{d} - F := \{D - F \mid D \in \mathfrak{d}\}$$

is a basepoint free linear system.

For example, let $X \subset \mathbb{P}^2$ be a smooth plane curve of degree d and $P \in X$ a point. Let

$$\tilde{\mathfrak{d}} = \{L \cdot X \mid L \subset \mathbb{P}^2 \text{ a line}\}$$

be the linear system associated to the embedding of X in $\mathbb{P}^2$, and

$$\mathfrak{d} = \{L \cdot X \mid L \subset \mathbb{P}^2 \text{ a line, } P \in L\} \subset \tilde{\mathfrak{d}}.$$

Then $\mathfrak{d}$ is not basepoint free, the fixed part $F = P$, and $\mathfrak{d}' = \mathfrak{d} - P$ corresponds to the morphism $f \colon X \to \mathbb{P}^1$ of degree $d-1$ given by projection from P. (Note: this is *not* an instance of 16.6 because the projection $p : \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$ is not defined at $P \in X$.)

17 The embedding criterion

Let X be a smooth projective curve and D a divisor on X.

Proposition 17.1. *(1) The complete linear system $|D|$ is basepoint free iff $l(D - P) = l(D) - 1$ for all $P \in X$.*

(2) Suppose $|D|$ is basepoint free. Then the associated morphism $\phi = \phi_{|D|}$ is an embedding iff $l(D-P-Q) = l(D)-2$ for all $P, Q \in X$ (we allow $P = Q$).

Proof. (1). Write $D = \sum n_P P$. Recall that we say $|D|$ is *basepoint free* if for all $P \in X$ there exists $D' \in |D|$ such that $P \notin \operatorname{Supp} D'$. Equivalently, there exists $f \in L(D)$ such that $\nu_P(f) = -n_P$ (writing $D' = D + (f)$). That is, in the exact sequence

$$0 \to L(D-P) \to L(D) \xrightarrow{\theta_P} k$$

the map θ_P is surjective. (Recall that θ_P is defined as follows: let t be a local parameter at P and write $f = a_{-n_P} t^{-n_P} + a_{-n_P+1} t^{-n_P+1} + \cdots$, then $\theta_P(f) = a_{-n_P}$.) This is equivalent to $l(D-P) = l(D) - 1$, as required.

(2). First suppose $P, Q \in X$, $P \neq Q$, and consider the exact sequence

$$0 \to L(D-P-Q) \to L(D) \xrightarrow{\theta_P \oplus \theta_Q} k \oplus k.$$

The map $\theta_P \oplus \theta_Q$ is surjective iff there exist $D_1, D_2 \in |D|$ such that $P \notin \operatorname{Supp} D_1$, $Q \in \operatorname{Supp} D_1$, and $P \in \operatorname{Supp} D_2$, $Q \notin \operatorname{Supp} D_2$. Recall that the elements of the linear system $|D|$ are the pullbacks $\phi^* H$ of hyperplanes $H \subset \mathbb{P}^N$. So there exist D_1, D_2 as above iff $\phi(P) \neq \phi(Q)$. Thus, $l(D-P-Q) = l(D) - 2$ iff $\phi(P) \neq \phi(Q)$.

Now suppose $P \in X$ and consider the map

$$0 \to L(D-2P) \to L(D) \xrightarrow{\theta_{2P}} k[t]/(t^2)$$

Here θ_{2P} is defined as follows: let t be a local parameter at P and write $f = a_{-n_P} t^{-n_P} + \cdots$, then $\theta_{2P}(f) = a_{-n_P} + a_{-n_P+1} t \in k[t]/(t^2)$. The map θ_{2P} is surjective iff there exists $D' \in |D|$ such that $n'_P = 1$ (where $D' = \sum n'_Q Q$). Equivalently, the hyperplane $H' \subset \mathbb{P}^N$ corresponding to D' is defined by an equation $(x = 0)$ near $\phi(P)$ such that $\nu_P(\phi^\sharp(x)) = 1$, that is, $\phi^\sharp(x)$ is a local parameter at $P \in X$. It follows that ϕ is locally an embedding near $P \in X$ iff $l(D-2P) = l(D) - 2$. Combining with the result of the previous paragraph gives the result. $\square$

Recall that for a curve $Y \subset \mathbb{P}^N$ in projective space, the *degree* of Y is the cardinality of $Y \cap H$ for $H \subset \mathbb{P}^N$ a general hyperplane.

Proposition 17.2. *Suppose $|D|$ is basepoint free. Let $\phi = \phi_{|D|} \colon X \to \mathbb{P}^N$ be the associated morphism, and $Y \subset \mathbb{P}^N$ the image of ϕ. Then* $\deg D = \deg \phi \cdot \deg Y$.

Proof. Recall that the elements of $|D|$ are the pullbacks $\phi^* H$ of hyperplanes $H \subset \mathbb{P}^N$. If H is general, then $Y \cap H$ consists of $\deg Y$ distinct points, and the fibre of ϕ over each point of $Y \cap H$ consists of $\deg \phi$ distinct points, where $\deg \phi$ is the degree of the surjective morphism of projective curves $\phi\colon X \to Y$. Thus $\deg D = \deg \phi^* H = \deg \phi \cdot \deg Y$. $\square$

Remark 17.3. Note that $Y = \phi(X) \subset \mathbb{P}^N$ may be a singular curve. We can define the degree of the morphism $\phi\colon X \to Y$ as $\deg \phi = [k(X)\colon k(Y)]$. Then over a smooth point of Y there are exactly $\deg \phi$ points (counting multiplicities), by the same proof as before.

In fact we have a factorisation $X \xrightarrow{\psi} Y^\nu \xrightarrow{\nu} Y$ of ϕ, where $\nu\colon Y^\nu \to Y$ is the *normalisation* of Y. Here Y^ν is a smooth projective curve and ν restricts to an isomorphism over the smooth locus of Y. Then $\deg \phi = \deg \psi$.

18 The Riemann–Roch theorem (first version)

Let X be a smooth projective curve, and D a divisor on X. The aim in this section is to compute the dimension $l(D)$ of $L(D)$.

An *adèle* r is a family $(r_P)_{P \in X}$ of elements of $k(X)$ such that $r_P \in \mathcal{O}_{X,P}$ for all but finitely many $P \in X$. The set R of adèles is a k-algebra. Given $f \in k(X)$, we define an associated adèle r by $r_P = f$ for all $P \in X$. This gives an inclusion $k(X) \subset R$.

Let $D = \sum n_P P$. We define

$$R(D) = \{r \in R \mid \nu_P(r_P) \geq -n_P \text{ for all } P \in X\},$$

a k-vector space. Then $R(D) \cap k(X) = L(D)$. We define

$$I(D) = \frac{R}{R(D) + k(X)},$$

a k-vector space. Then, by construction, we have an exact sequence

$$0 \to L(D) \to k(X) \to R/R(D) \to I(D) \to 0.$$

An element $\bar{r} = (\bar{r}_P) \in R/R(D)$ can be described as follows: at each point $P \in X$, $\bar{r}_P$ is given by a *Laurent tail*

$$\bar{r}_P = a_\nu t^\nu + a_{\nu+1} t^{\nu+1} + \cdots + a_{-n_P-1} t^{-n_P-1}$$

where t is a local parameter at $P \in X$, $\nu \in \mathbb{Z}$, $\nu \leq -n_P$, and $a_i \in k$, and only finitely many of the $\bar{r}_P$ are nonzero.

Example 18.1. Let $X = \mathbb{P}^1$. Then $I(0) = 0$. Indeed, an element $\bar{r} \in R/R(0)$ is a finite collection of Laurent tails as above, where $n_P = 0$ for all P. Choose affine coordinate $x = X_1/X_0$ on $\mathbb{P}^1$ such that the points $P_1, \ldots, P_k$ such that $\bar{r}_P \neq 0$ lie in the affine piece $\mathbb{A}^1_x = (X_0 \neq 0) \subset \mathbb{P}^1$, with x coordinates $\alpha_1, \ldots, \alpha_k$. Then $x - \alpha_i$ is a local parameter at P_i, and we can write

$$\bar{r}_{P_i} = f_i = a_{\nu_i,i}(x - \alpha_i)^{\nu_i} + \cdots + a_{-1,i}(x - \alpha)^{-1}.$$

Define $f = \sum f_i \in k(X)$, then f has Laurent tail $\bar{r}_{P_i}$ at P_i and is regular elsewhere, that is, $f \mapsto \bar{r} \in R/R(0)$. Hence $I(0) = 0$ as claimed.

Example 18.2. Let X be a smooth projective curve which is not isomorphic to $\mathbb{P}^1$. Then there does not exist a rational function $f \in k(X)$ such that f has a simple pole at a point $P \in X$ and is regular elsewhere. So in this case $k(X) \to R/R(0)$ is not surjective, that is, $I(0) \neq 0$.

Lemma 18.3. *Suppose $D \leq D'$. Then there is a natural surjection $I(D) \to I(D')$, and the kernel has dimension*

$$(\deg D' - l(D')) - (\deg D - l(D)).$$

Proof. Recall that $I(D) = R/(R(D) + k(X))$ and $R(D) \subseteq R(D')$ for $D \leq D'$ by definition, so $I(D)$ surjects onto $I(D')$. Consider the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & k(X)/L(D) & \longrightarrow & R/R(D) & \longrightarrow & I(D) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & k(X)/L(D') & \longrightarrow & R/R(D') & \longrightarrow & I(D') & \longrightarrow & 0
\end{array}$$

The vertical arrows are surjective, so the kernels form an exact sequence

$$0 \to L(D')/L(D) \to R(D')/R(D) \to K \to 0$$

where $K = \ker(I(D) \to I(D'))$. Finally, observe that $R(D')/R(D)$ has dimension $\deg D' - \deg D$. Indeed, writing $D = \sum n_P P$ and $D' = \sum n'_P P$, an element $\bar{r} \in R(D')/R(D)$ is given by Laurent tails

$$\bar{r}_P = a_{-n'_P} t^{-n'_P} + a_{-n'_P+1} t^{-n'_P+1} + \cdots + a_{-n_P-1} t^{-n_P-1},$$

where t denotes a local parameter at P. So

$$\dim R(D')/R(D) = \sum_P (n'_P - n_P) = \deg D' - \deg D$$

as claimed. We deduce the formula for $\dim K$ stated above. $\square$

Lemma 18.4. *Let X be a smooth projective curve. Then there exists $M \in \mathbb{N}$ such that* $\deg D - l(D) \leq M$ *for all divisors D on X.*

Proof. First observe that if $D \leq E$ then $\deg D - l(D) \leq \deg E - l(E)$. Indeed, by induction, we may assume $E = D + P$, and $l(D+P) \leq l(D)+1$ by the exact sequence

$$0 \to L(D) \to L(D+P) \to k.$$

Let $f \in k(X)$ be a nonconstant rational function and

$$F = (1 : f) \colon X \to \mathbb{P}^1$$

the corresponding morphism. Let $A = F^*(0 : 1)$. So A is the divisor of degree $d = \deg F$ given by the sum of the poles of f (with multiplicities).

Let D be a divisor on X. We claim that there exists a linearly equivalent divisor D' such that $D' \leq nA$ for some $n \in \mathbb{N}$. Indeed, write $D = \sum n_P P$, and define

$$h = \prod_{P \in S} (f - f(P))^{n_P}.$$

where $S = \{P \in X \mid n_P > 0, f(P) \neq \infty\}$. Then $(h) \geq D - nA$ for some $n \in \mathbb{N}$, that is, $D' := D - (h) \leq nA$, as required.

We next establish the result for the divisors $D = nA$, $n \in \mathbb{N}$. The morphism $F = (1 : f)$ corresponds to the field extension $k(f) \subset k(X)$ of degree d. Pick a basis $g_1, \ldots, g_d$ for $k(X)$ over $k(f)$. Then, by the construction of the previous paragraph, there exist polynomials $p_i(T) \in k[T]$ such that $p_i(f)g_i \in L(n_0 A)$ for all i, some $n_0 \in \mathbb{N}$. Indeed, we let $D = -(g_i)$ and define $h = p_i(f)$ as above, then $(hg_i) = -D' \geq -n_i A$ for some n_i, and we let $n_0 = \max\{n_i\}$. Then $f^j p_i(f) g_i \in L(nA)$ for each $1 \leq i \leq d$ and $0 \leq j \leq n - n_0$. Moreover, these functions are linearly independent over k (because $g_1, \ldots, g_d$ are linearly independent over $k(f)$). So $l(nA) \geq (n - n_0 + 1)d = \deg nA - (n_0 - 1)d$, that is, $\deg nA - l(nA) \leq M$ where $M := (n_0 - 1)d$.

Now we combine our results. Given D, let $D' \sim D$ and $D' \leq nA$. Then

$$\deg D - l(D) = \deg D' - l(D') \leq \deg nA - l(nA) \leq M.$$

□

Lemma 18.5. *The k-vector space $I(D)$ is finite dimensional.*

Proof. By Lem. 18.4 there exists a divisor D_0 on X such that $\deg D_0 - l(D_0)$ is maximal. We claim that $I(D_0) = 0$. Indeed, otherwise let $0 \neq r = (r_P) \in I(D_0) = R/(R(D_0) + k(X))$. There exists $D' \geq D_0$ such that r lies in the kernel of the surjection $I(D_0) \to I(D')$. (Indeed, just pick $D' = \sum n'_P P$ such that $\nu_P(r_P) \geq -n'_P$ for all $P \in X$, that is , $r \in R(D')$.) So $\deg D' - l(D') > \deg D_0 - l(D_0)$ by Lem. 18.3, a contradiction.

If $D \leq D'$ we have a surjection $I(D) \to I(D')$ with finite dimensional kernel by Lem. 18.3. Thus $I(D)$ is finite dimensional iff $I(D')$ is so. Since $I(D_0)$ is finite dimensional, we deduce that $I(D)$ is finite dimensional for every divisor D. $\square$

Write $i(D) = \dim_k I(D)$.

Theorem 18.6 (Riemann–Roch v1). $l(D) - i(D) = 1 - i(0) + \deg D$

Proof. By Lem. 18.3, for $D \leq D'$ we have

$$(\deg D' - l(D')) - (\deg D - l(D)) = i(D) - i(D'),$$

that is,

$$l(D) - i(D) - \deg D = l(D') - i(D') - \deg D'.$$

We deduce that, for any divisor D,

$$l(D) - i(D) - \deg D = l(0) - i(0) - \deg 0 = 1 - i(0).$$

$\square$

19 The canonical divisor

Let X be a smooth projective curve and write $K = k(X)$. Let $\omega \in \Omega_{K/k}$ be a (nonzero) rational 1-form on X. We define a divisor

$$(\omega) = \sum_{P \in X} \nu_P(\omega) P$$

where $\nu_P(\omega)$ is the order of vanishing of ω at P. (Recall, we can write $\omega = f dt$ where t is a local parameter at P and $f \in K$, then $\nu_P(\omega) := \nu_P(f)$.)

Note that, if ω' is another rational 1-form, then $\omega' = f\omega$ for some $f \in K$ (because $\Omega_{K/k}$ is a 1-dimensional K-vector space). Then $(\omega') = (f) + (\omega)$. So the linear equivalence class of (ω) is well defined. It is called the *canonical divisor class* and denoted K_X.

Proposition 19.1. *($k = \mathbb{C}$) $\deg K_X = 2g - 2$ where g is the genus of X.*

Proof. Pick $f \in k(X) \setminus k$ and let $F = (1 : f) \colon X \to \mathbb{P}^1$ be the corresponding morphism. The Riemann-Hurwitz formula gives

$$2g - 2 = d(-2) + \sum_{P \in X} (e_P - 1),$$

where $d = \deg F$.

Let $\omega = df \in \Omega_{K/k}$. Let $P \in X$ be a point, t a local parameter at P, and write $e = e_P$. If $F(P) \neq (0 : 1)$, then $f = a_e t^e + a_{e+1} t^{e+1} + \cdots$ and $df = (e a_e t^{e-1} + \cdots) dt$, so $\nu_P(df) = e - 1$. If $F(P) = (0 : 1)$, then $f = a_{-e} t^{-e} + \cdots$ and $df = (-e a_{-e} t^{-e-1} + \cdots) dt$, so $\nu_P(df) = -e - 1$. So

$$\deg(\omega) = \sum_{f(P) \neq \infty} (e_P - 1) + \sum_{f(P) = \infty} -(e_P + 1)$$

$$= \sum_{P \in X} (e_P - 1) - 2 \sum_{f(P) = \infty} e_P = \sum_{P \in X} (e_P - 1) - 2d.$$

Thus $\deg(\omega) = 2g - 2$ by the Riemann-Hurwitz formula. □

20 Serre duality

Let X be a smooth projective curve and $K = k(X)$. For D a divisor on X, we define

$$\Omega(D) = \{\omega \in \Omega_{K/k} \mid (\omega) + D \geq 0\},$$

a k-vector space.

Note that, if ω_0 is a nonzero rational 1-form and we write $K_X = (\omega_0)$, then we have an isomorphism

$$L(K_X + D) \xrightarrow{\sim} \Omega(D), \quad f \mapsto f\omega_0.$$

We define a k-bilinear pairing

$$\langle\, , \rangle \colon \Omega_{K/k} \times R \to k, \quad \langle \omega, r \rangle = \sum_{P \in X} \mathrm{Res}_P(r_P \omega).$$

(Note immediately that the sum in the definition is finite.) The pairing $\langle\, , \rangle$ satisfies the following properties.

(1) $\langle \omega, r \rangle = 0$ if $r \in k(X)$.

(2) $\langle \omega, r \rangle = 0$ if $\omega \in \Omega(-D)$ and $r \in R(D)$ for some divisor D.

(3) $\langle f\omega, r \rangle = \langle \omega, fr \rangle$ for $f \in k(X)$.

Indeed, (1) is a restatement of the residue theorem, (2) holds because in this case $r_P\omega$ is regular at P for each $P \in X$, and (3) is clear.

Recall that $I(D) = R/(R(D) + k(X))$. So by properties (1) and (2) above we obtain a k-bilinear pairing

$$\langle \, , \, \rangle \colon \Omega(-D) \times I(D) \to k.$$

Theorem 20.1 (Serre duality). *The pairing* $\langle \, , \, \rangle \colon \Omega(-D) \times I(D) \to k$ *is perfect. That is, the induced map*

$$\Omega(-D) \to I(D)^*, \quad \omega \mapsto \langle \omega, \cdot \rangle$$

is an isomorphism (here $I(D)^* = \operatorname{Hom}_k(I(D), k)$*, the dual space.)*

In particular, $l(K_X - D) = i(D)$.

Proof. Write $J(D) = I(D)^*$. Recall that if $D \leq D'$ then $I(D) \twoheadrightarrow I(D')$, so $J(D') \subset J(D)$. Let $J = \bigcup_D J(D)$. (Equivalently, J is the subspace of the dual $\operatorname{Hom}_k(R/k(X), k)$ of the infinite dimensional k-vector space $R/k(X)$ consisting of linear maps which vanish on $R(D)/k(X)$ for some D.)

For $f \in K$ and $\alpha \in J$, we define $(f\alpha)(r) = \alpha(fr)$ for $r \in R$. Then $(f\alpha) \in J$. Indeed, $f\alpha(k(X)) = \alpha(f \cdot k(X)) = \alpha(k(X)) = 0$. Also, if $\alpha \in J(D)$ and $f \in L(\Delta)$ then $f\alpha \in J(D - \Delta)$ (because if $r \in R(D - \Delta)$ then $fr \in R(D)$ and so $(f\alpha)(r) = \alpha(fr) = 0$). This defines the structure of a K-vector space on J.

Lemma 20.2. $\dim_K J = 1$

Proof. Clearly $J \neq 0$. (For example, $l(D) - i(D) = 1 - i(0) + \deg D$ by the Riemann-Roch theorem v1, so $i(D) > 0$ if $\deg D < -1$). Suppose there exist $\alpha_1, \alpha_2 \in J$ which are linearly independent over K. Then $\alpha_1, \alpha_2 \in J(D)$ for some D. As observed above scalar multiplication defines a map

$$L(\Delta) \times J(D) \to J(D - \Delta).$$

Let $P \in X$ be a point. We obtain an inclusion

$$L(nP)^{\oplus 2} \subset J(D - nP), \quad (f_1, f_2) \mapsto f_1\alpha_1 + f_2\alpha_2$$

for all $n \geq 0$. Recall (Lem. 18.4) that $\deg D - l(D) \leq M$ for all D, some fixed $M \geq 0$. Thus $l(nP) \geq n - M$. Also

$$\dim_k J(D - nP) = i(D - nP) = l(D - nP) - (1 - i(0) + \deg(D - nP))$$

$$= n + l(D - nP) - (1 - i(0) + \deg D)$$

by the Riemann Roch theorem. Combining,

$$2n - 2M \leq n + l(D - nP) - (1 - i(0) + \deg D).$$

But $l(D - nP) = 0$ for $n \gg 0$ so we obtain a contradiction. $\square$

Lemma 20.3. *Let $\theta \colon \Omega_{K/k} \to J$ be the K-linear map $\theta(\omega) = \langle \omega, \cdot \rangle$. Then $\theta(\Omega(-D)) \subseteq J(D)$. Conversely, if $\theta(\omega) \in J(D)$ then $\omega \in \Omega(-D)$.*

Proof. Suppose $\theta(\omega) \in J(D)$ and $\omega \notin \Omega(-D)$. Then there exists a point $P \in X$ such that $\nu_P(\omega) < n_P$ (writing $D = \sum n_Q Q$). Let t be a local parameter at P and $n = \nu_P(\omega) + 1 \leq n_P$. Define an adèle $r = (r_Q)$ by $r_P = 1/t^n$ and $r_Q = 0$ for $Q \neq P$. Then $r \in R(D)$ and $\langle \omega, r \rangle = \mathrm{Res}_P(r_P \omega) \neq 0$. So $\omega \notin J(D)$, a contradiction. $\square$

We can now finish the proof of the Serre duality theorem. The K-linear map $\theta \colon \Omega_{K/k} \to J$ is an isomorphism of K-vector spaces (because θ is nonzero and $\Omega_{K/k}$ and J have dimension 1 over K). In particular, the induced map $\theta_D \colon \Omega(-D) \to J(D)$ is injective, and θ_D is surjective by 20.3. So θ_D is an isomorphism of k-vector spaces, as required. $\square$

21 Riemann–Roch, final version

Let X be a smooth projective curve and D a divisor on X. Then

$$l(D) - i(D) = 1 - i(0) + \deg D,$$

by the Riemann–Roch theorem v1, and $i(D) = l(K_X - D)$ by Serre duality, so

$$l(D) - l(K_X - D) = 1 - i(0) + \deg D.$$

Now set $D = K_X$. We obtain

$$l(0) - l(K_X) = 1 - i(0) + \deg K_X,$$

that is,

$$2l(K_X) - 2 = \deg K_X \tag{1}$$

since $l(0) = 1$ and $i(0) = l(K_X)$ by Serre duality. Recall that, if $k = \mathbb{C}$, then $\deg K_X = 2g - 2$ where g is the genus of X (number of holes). So $l(K_X) = i(0) = g$ by (1). In general, we *define* the genus g of X by $g = l(K_X) = i(0)$, then $\deg K_X = 2g - 2$ by (1).

Theorem 21.1. *Let X be a smooth projective curve of genus g over an algebraically closed field k. Then $l(K_X) = \dim_k \Omega_X(X) = i(0) = g$ and $\deg K_X = 2g-2$.*

□

Theorem 21.2. *(Riemann–Roch theorem, v2) Let X be a smooth projective curve of genus g over an algebraically closed field k and D a divisor on X. Then*

$$l(D) - l(K_X - D) = 1 - g + \deg D.$$

□

The Riemann–Roch theorem is a powerful tool in the theory of algebraic curves.

Corollary 21.3. *If $\deg D \geq 2g-1$ then $l(D) = 1 - g + \deg D$.*

Proof. We have $\deg(K_X - D) = 2g - 2 - \deg D < 0$ so $l(K_X - D) = 0$. □

Corollary 21.4. *If $g = 0$ then $X \simeq \mathbb{P}^1$*

Proof. Let $P \in X$ be a point. Then $l(P) = 2$ by 21.3. Let $f \in L(P) \setminus k$, then $(1 : f)\colon X \to \mathbb{P}^1$ is an isomorphism. □

Corollary 21.5. *If $\deg D \geq 2g+1$ then the linear system $|D|$ is basepoint free and the associated morphism*

$$\phi_{|D|}\colon X \to \mathbb{P}^{l(D)-1}$$

is an embedding. Note also that $l(D) = 1 - g + \deg D$ and the image has degree $\deg D$ as a subvariety of projective space.

Proof. Recall that $|D|$ is basepoint free and $\phi_{|D|}$ is an embedding iff

$$l(D - P - Q) = l(D) - 2$$

for all $P, Q \in X$. Since $\deg D \geq 2g+1$, this follows from 21.3. □

Example 21.6. Let $\deg D = 2g+1$, then we obtain an embedding $\phi_{|D|}$ of X as a curve of degree $2g+1$ in $\mathbb{P}^{g+1}$. For example if $g = 1$ we obtain an embedding as a plane curve of degree 3. (Conversely, by the genus formula $g = (d-1)(d-2)/2$ for plane curves of degree d, every plane cubic has genus 1).

Corollary 21.7. *Assume X is not isomorphic to $\mathbb{P}^1$. Then the canonical linear system $|K_X|$ is basepoint free and the associated morphism $\phi_{|K_X|}$ is an embedding iff X is not hyperelliptic.*

Proof. The linear system $|K_X|$ is basepoint free iff $l(K_X - P) = l(K_X) - 1$ for all $P \in X$. Now $l(K_X) = g$,

$$l(K_X - P) - l(P) = 1 - g + \deg(K_X - P) = g - 2$$

by the Riemann-Roch theorem, and $l(P) = 1$ because X is not isomorphic to $\mathbb{P}^1$. So $|K_X|$ is basepoint free.

Similarly, $\phi_{|K_X|}$ is an embedding iff $l(K_X - P - Q) = l(K_X) - 2$ for all $P, Q \in X$, and

$$l(K_X - P - Q) - l(P + Q) = 1 - g + \deg(K_X - P - Q) = g - 3$$

by Riemann–Roch. So $\phi_{|K_X|}$ is an embedding iff $l(P+Q) = 1$ for all $P, Q \in X$. If $l(P+Q) > 1$ then $l(P+Q) = 2$ and $\phi_{|P+Q|}\colon X \to \mathbb{P}^1$ is a morphism of degree 2, so X is hyperelliptic. (Conversely, if X is hyperelliptic, let $f\colon X \to \mathbb{P}^1$ be a morphism of degree 2, and $P + Q$ a fibre of f, then $l(P+Q) = 2$). $\square$

It is important to note that $\phi = \phi_{|K_X|}\colon X \to \mathbb{P}^{g-1}$ is uniquely determined up to composition by $\mathrm{PGL}_g(k) = \mathrm{Aut}(\mathbb{P}^{g-1})$ by the curve X. Indeed, ϕ is determined by a choice of basis of $L(K_X) = \Omega_X(X)$. In particular, if $\theta\colon X \xrightarrow{\sim} X$ is an automorphism of X, then there exists a unique $\tilde{\theta} \in \mathrm{PGL}_g(k)$ such that the following diagram commutes.

$$\begin{array}{ccc} X & \xrightarrow{\phi} & \mathbb{P}^{g-1} \\ {\scriptstyle\theta}\downarrow & & \downarrow{\scriptstyle\tilde{\theta}} \\ X & \xrightarrow{\phi} & \mathbb{P}^{g-1} \end{array}$$

Informally, ϕ does not "break symmetry". Explicitly, ϕ is defined by picking a basis $\omega_0, \ldots, \omega_{g-1}$ of $\Omega_X(X)$, and writing $\phi = (\omega_0 : \ldots : \omega_{g-1})$. (Note that the ratio of two 1-forms is a rational function, so this makes sense). Then the composition $\phi \circ \theta$ is defined by the pullbacks $\theta^*(\omega_i)$ of these forms. (Here, for $f\colon Y \to Z$ a morphism of smooth varieties, and ω a regular 1-form on Z, the pullback $f^*\omega$ of ω is defined locally as follows: write $\omega = \sum g_i dh_i$, then $f^*\omega = \sum f^\sharp(g_i) d(f^\sharp(h_i))$.) Let A be the matrix of the isomorphism $\theta^*\colon \Omega_X(X) \xrightarrow{\sim} \Omega_X(X)$ with respect to the basis $\{\omega_0, \ldots, \omega_{g-1}\}$. Then the map $\tilde{\theta}$ above is given by $A^T \in \mathrm{PGL}_g(k)$.

Next we compare with the case of a curve X of genus 1 over $k = \mathbb{C}$. (Note that in this case $K_X = 0$ so the canonical map $\phi_{|K_X|}$ is the constant map $X \to \text{pt.}$) Recall that X^{an} is a complex torus $\mathbb{C}/L$, where $L \simeq \mathbb{Z}^2$ and L spans $\mathbb{C}$ as an $\mathbb{R}$-vector space. In particular the automorphism group of X is transitive (because X is a group and acts on itself by translation). Recall that if D is a divisor of degree 3 on X then $|D|$ defines an embedding $\phi\colon X \to \mathbb{P}^2$ of X as a plane cubic. If we let $D = 3P$ then $\phi(P)$ is an inflection point (or flex) of $\phi(X) \subset \mathbb{P}^2$. Indeed, the divisor $D = 3P \in |D|$ corresponds to the pullback ϕ^*H of a hyperplane $H \subset \mathbb{P}^2$. Then H is a line which is tangent to $\phi(X)$ at $\phi(P)$ with order of contact equal to 3, so $\phi(P)$ is a flex of $\phi(X)$. There are only finitely many flexes of $\phi(X)$, and any automorphism θ of X which extends to an automorphism $\tilde{\theta}$ of $\mathbb{P}^2$ must permute the flexes. Hence only a finite subgroup of the subgroup $X \subset \text{Aut}(X)$ of translations of X extend to automorphisms of $\mathbb{P}^2$. (Note also that $\text{Aut}(X)/X$ is finite, equal to $\mathbb{Z}/2\mathbb{Z}$, $\mathbb{Z}/4\mathbb{Z}$, or $\mathbb{Z}/6\mathbb{Z}$.)

Theorem 21.8. *Let X be a smooth projective curve of genus $g \geq 2$. Then* $\text{Aut}(X)$ *is finite.*

Proof. (Sketch) If X is hyperelliptic then $\phi_{|K_X|}\colon X \to \mathbb{P}^{g-1}$ is the composition of the degree 2 map $f\colon X \to \mathbb{P}^1$ and the embedding

$$\mathbb{P}^1 \to \mathbb{P}^{g-1}, \quad (X_0 : X_1) \mapsto (X_0^{g-1} : \cdots : X_1^{g-1}).$$

This follows from our explicit computation of $\Omega_X(X)$ for a hyperelliptic curve. (In particular, this shows that the map f is uniquely determined by X up to composition with an element of $\text{PGL}_2(k) = \text{Aut}(\mathbb{P}^1)$.) Now, if θ is an automorphism of X, we obtain an automorphism $\tilde{\theta}$ of $\mathbb{P}^{g-1}$ compatible with θ and $\phi = \phi_{K_X}$ as above. Thus $\tilde{\theta}$ restricts to an isomorphism $\mathbb{P}^1 \xrightarrow{\sim} \mathbb{P}^1$ which permutes the $2g+2$ branch points of f. Also $\tilde{\theta}$ determines θ up to $\mathbb{Z}/2\mathbb{Z}$ (generated by the deck transformation for f). An automorphism of $\mathbb{P}^1$ which fixes 3 distinct points is equal to the identity. So, we obtain an exact sequence of groups

$$0 \to \mathbb{Z}/2\mathbb{Z} \to \text{Aut}(X) \to S_{2g+2}.$$

In particular, $\text{Aut}(X)$ is finite.

Now suppose X is not hyperelliptic. So $\phi = \phi_{|K_X|}$ is an embedding. We have an inclusion $\text{Aut}(X) \subset \text{PGL}_g(k)$ given (in the notation used above) by $\theta \mapsto \tilde{\theta}$. Note that $\text{PGL}_g(k)$ is a quasiprojective variety over k, and $\text{Aut}(X) \subset \text{PGL}_g(k)$ is Zariski closed (because a matrix $A \in \text{PGL}_g(k)$ preserves the subvariety $\phi(X) \subset \mathbb{P}^{g-1}$ iff certain homogeneous polynomials in the entries of A vanish). In particular $\mathrm{Aut}(X)$ is finite iff it has dimension 0. As in differential geometry, a tangent vector to $\mathrm{Aut}(X)$ at the identity $e \in \mathrm{Aut}(X)$ corresponds to an (algebraic) vector field on X. We show that there are no vector fields on X for $g \geq 2$, so $e \in \mathrm{Aut}(X)$ is an isolated point, and $\mathrm{Aut}(X)$ has dimension 0 as required.

We define vector fields in algebraic geometry as follows. Let X be a smooth variety and $K = k(X)$. Let $T_{K/k} = \mathrm{Hom}(\Omega_{K/k}, K)$, the rational vector fields. For $P \in X$ a point, let $T_{X,P} = \mathrm{Hom}_{\mathcal{O}_{X,P}}(\Omega_{X,P}, \mathcal{O}_{X,P})$, the regular vector fields at $P \in X$. For $U \subset X$ an open subset, let

$$T_X(U) = \bigcap_{P \in X} T_{X,P} \subset T_{K/k},$$

the regular vector fields on U. If $P \in X$ is a point and $t_1, \ldots, t_n$ is a system of local parameters at $P \in X$ (where $n = \dim X$), then an element $v \in T_{X,P}$ is of the form $f_1 \frac{d}{dt_1} + \cdots + f_n \frac{d}{dt_n}$ where $f_i \in \mathcal{O}_{X,P}$ and $\frac{d}{dt_i}(dt_j) = \delta_{ij}$.

Now suppose X is a smooth projective curve. Then we have an identification $T_X(X) = L(-K_X)$. Indeed, let $\omega \in \Omega_{K/k}$ be a 1-form and write $K_X = (\omega)$. Then a rational vector field $Y \in T_{K/k}$ corresponds to a rational function $f \in K$ via $Y(\omega) = f$, and Y is regular iff $f \in L(-K_X)$. (To see this, let $K_X = \sum n_P P$. We work locally at $P \in X$, let t be a local parameter at P, and write $Y = g\frac{d}{dt}$, $\omega = hdt$, then $Y(\omega) = gh = f$. Then Y is regular at P iff $\nu_P(g) \geq 0$, and $\nu_P(g) = \nu_P(f) - \nu_P(h) = \nu_P(f) - n_P$. So Y is regular iff $(f) - K_X \geq 0$, as required). Now $\deg K_X = 2g - 2 > 0$ because $g \geq 2$, so $L(-K_X) = 0$. This completes the proof. $\square$

Remark 21.9. The vector space $T_X(X)$ of regular vector fields on a smooth algebraic variety X corresponds to the tangent space at the identity to the automorphism group of X. If X is a smooth projective curve of genus g then $\dim_k T_X(X) = l(-K_X) = 3, 1, 0$ for $g = 0, 1, \geq 2$ respectively. More precisely, if $g = 0$ then $X \simeq \mathbb{P}^1$ and $\mathrm{Aut}(X) \simeq \mathrm{PGL}_2(k)$, if $g = 1$ then X is a complex torus and the connected component of $\mathrm{Aut}(X)$ containing the identity is a copy of X (corresponding to translations), and if $g \geq 2$ then $\mathrm{Aut}(X)$ is finite.

22 Curves of low genus

Here we give explicit descriptions of curves of low genus. Let X be a smooth projective curve of genus g. Recall that if $g = 0$ then $X \simeq \mathbb{P}^1$ and if $g = 1$ then X is isomorphic to a plane cubic.

Proposition 22.1. *If $g = 2$ then X is hyperelliptic.*

Proof. Since $l(K_X) = g = 2$ and $\deg K_X = 2g - 2 = 2$ the canonical map ϕ_{K_X} is a degree 2 morphism $X \to \mathbb{P}^1$. $\square$

Let M_g denote the moduli space of smooth projective curves of genus $g \geq 2$ over an algebraically closed field k. This is an algebraic variety over k whose points correspond to isomorphism classes of smooth curves of genus g. Moreover the geometry of M_g encodes how curves vary in families — roughly speaking, a morphism $f : S \to M_g$ of algebraic varieties corresponds to a morphism $X \to S$ where, for each $s \in S$, the fibre X_s is a smooth curve of genus g, and $f(s) = [X_s]$, the point of M_g given by the isomorphism class of X_s. (The construction of the variety M_g is outside the scope of this course, we will just assume its existence and describe some small examples.) It is known that the variety M_g is irreducible of dimension $3g - 3$ and has only quotient singularities.

If X is a curve of genus 2 then X is hyperelliptic and the degree 2 map $f : X \to \mathbb{P}^1$ is branched over $2g + 2 = 6$ points. (Recall also that for a hyperelliptic curve of genus $g \geq 2$ the map f is uniquely determined up to composition with an element of PGL(2) — because f can be recovered from ϕ_{K_X}). So M_2 is the space of sets of 6 unordered distinct points on $\mathbb{P}^1$, modulo PGL(2). A set of 6 points on $\mathbb{P}^1$ is the zero locus of a homogeneous polynomial $F_6(X_0, X_1)$ of degree 6, determined up to multiplication by a nonzero scalar. So the space of sets of 6 points on $\mathbb{P}^1$ is a $\mathbb{P}^6$ (with homogeneous coordinates given by the coefficients of F_6). The locus $\Delta \subset \mathbb{P}^6$ corresponding to polynomials with repeated roots is a hypersurface with equation the discriminant of the polynomial F_6. We deduce that

$$M_2 = (\mathbb{P}^6 \setminus \Delta)/\operatorname{PGL}(2).$$

In particular $\dim M_2 = \dim \mathbb{P}^6 - \dim \operatorname{PGL}(2) = 6 - 3 = 3$. This agrees with the formula $M_g = 3g - 3$ stated above.

Proposition 22.2. *If $g = 3$ then either X is isomorphic to a plane quartic or X is hyperelliptic.*

Proof. We already know that ϕ_{K_X} is an embedding if X is not hyperelliptic. In this case $l(K_X) = g = 3$ and $\deg K_X = 2g - 2 = 4$ imply that ϕ_{K_X} realises X as a plane quartic. $\square$

We will see later (Prop. 23.1) that if $X \subset \mathbb{P}^2$ is a smooth plane quartic then the linear system δ given by hyperplane sections of X in this embedding is equal to the canonical linear system $|K_X|$. That is, X is embedded via ϕ_{K_X}. Assuming this, we can describe M_3 as follows. The moduli space M_3 is a disjoint union $A \sqcup B$ where A is the space of smooth plane quartics and B is the space of hyperelliptic curves of genus 3. As above, we can describe A as a quotient $(\mathbb{P}^{14} \setminus \Delta)/\operatorname{PGL}(3)$ where the homogeneous coordinates on $\mathbb{P}^{14}$ are the coefficients of the equation of the plane quartic, and $\Delta \subset \mathbb{P}^{14}$ is the locus of singular curves (an irreducible hypersurface), and $B = (\mathbb{P}^8 \setminus \Delta)/\operatorname{PGL}(2)$ (a hyperelliptic curve of genus $g = 3$ is branched over $2g + 2 = 8$ points on $\mathbb{P}^1$). In particular $\dim A = 14 - \dim \operatorname{PGL}(3) = 14 - 8 = 6$ and $\dim B = 5$. In fact $A \subset M_3$ is open and its complement $B \subset M_3$ is a closed subvariety of codimension 1.

Proposition 22.3. *If $g = 4$ then either X is the intersection of a quadric hypersurface and a cubic hypersurface in $\mathbb{P}^3$, or X is hyperelliptic.*

Proof. Assume X is not hyperelliptic. Then ϕ_{K_X} is an embedding $X \hookrightarrow \mathbb{P}^3$ of degree $2g - 2 = 6$.

In general, suppose given a morphism $\phi \colon X \to \mathbb{P}^N$ corresponding to a basepoint free linear system $\delta \subset |D|$. Then $\phi = (f_0 : \cdots : f_N)$ where $f_0, \ldots, f_N$ is a basis of the subspace of $L(D)$ corresponding to δ. Let $S = k[X_0, \ldots, X_N]$, the homogeneous coordinate ring of $\mathbb{P}^N$, and $S_n \subset S$ the k-vector space of homogeneous polynomials of degree n (so $S = \oplus_{n \geq 0} S_n$). We have a map of k-vector spaces

$$S_1 \to L(D), \quad X_i \mapsto f_i$$

which induces maps

$$\theta_n \colon S_n \to L(nD)$$

for each $n \geq 1$. The kernel of θ_n is the space of homogeneous polynomials of degree n vanishing on $\phi(X) \subset \mathbb{P}^N$.

Now consider our example. The map θ_1 is an isomorphism (because $\phi = \phi_{K_X}$ is defined by a complete linear system). The map θ_2 has nontrivial kernel because

$$\dim S_2 = 10 > \dim L(2K_X) = 1 - g + \deg(2K_X) = 3g - 3 = 9.$$

So $X \subset \mathbb{P}^3$ is contained in a quadric hypersurface $Q = (F_2 = 0) \subset \mathbb{P}^3$. Note that Q is irreducible (because X is not contained in a hyperplane) but may be singular. Now consider the map θ_2. We compute $\dim S_3 = 20$ and $l(3K_X) = 15$. Thus $\dim \ker \theta_2 \geq 5$, so $\ker \theta_2 \supsetneq \langle X_0 F_2, \ldots, X_3 F_2 \rangle$, and there exists an irreducible cubic G_3 such that $X \subset (F_2 = G_3 = 0) \subset \mathbb{P}^3$. Now since $\deg X = 6$ it follows that $X = (F_2 = G_3 = 0)$. $\square$

23 Plane curves

Here we use Riemann–Roch to deduce some properties of plane curves. In particular, we show that a plane curve of degree $d \geq 4$ is never hyperelliptic.

Proposition 23.1. *Let $X \subset \mathbb{P}^2$ be a smooth plane curve of degree d and $\delta \subseteq |D|$ the linear system on X associated to the embedding. Then $K_X = (d-3)D$, and the genus of X is given by $g = (d-1)(d-2)/2$. Moreover, the map*

$$S_{d-3} \to L((d-3)D) = L(K_X) = \Omega_X(X)$$

is an isomorphism of k-vector spaces. Equivalently, every divisor in the canonical linear system $|K_X|$ is the intersection $X \cdot C$ of X with a unique curve $C \subset \mathbb{P}^2$ of degree $d-3$.

Proof. Recall that δ is the set of hyperplane sections of X in the given embedding.

We just compute explicitly. Let $X = (F(X_0, X_1, X_2) = 0) \subset \mathbb{P}^2$. The affine piece $(X_0 \neq 0)$ of X is

$$U_0 = (f(x_1, x_2) = 0) \subset \mathbb{A}^2_{x_1,x_2},$$

where $x_1 = X_1/X_0$, $x_2 = X_2/X_0$, and $f(x_1, x_2) = F(1, x_1, x_2)$. The rational 1-form

$$\omega = \frac{dx_1}{\frac{\partial f}{\partial x_2}} = -\frac{dx_2}{\frac{\partial f}{\partial x_1}}$$

is regular and nowhere zero on U_0. Indeed, for all $P \in U_0$, either $\frac{\partial f}{\partial x_1}(P) \neq 0$ and x_2 is a local parameter or $\frac{\partial f}{\partial x_2}(P) \neq 0$ and x_1 is a local parameter, by smoothness of X. Now we compute the divisor (ω) using the other charts. We may assume that $(0:1:0) \notin X$, so X is covered by the affine pieces $U_0 = (X_0 \neq 0)$ and $U_2 = (X_2 \neq 0)$. We have

$$U_2 = (g(y_0, y_1) = 0) \subset \mathbb{A}^2_{y_0,y_1}$$

where $y_0 = X_0/X_2$, $y_1 = X_1/X_2$, and $g(y_0, y_1) = F(y_0, y_1, 1)$. Now $x_1 = y_1/y_0$ and $x_2 = y_0^{-1}$, so $y_0 = x_2^{-1}$, $y_1 = x_1/x_2$, and $f(x_1, x_2) = y_0^{-d} g(y_0, y_1)$. We compute

$$\omega = -\frac{dx_2}{\frac{\partial f}{\partial x_1}} = \frac{y_0^{-2} dy_0}{y_0^{-d} \frac{\partial g}{\partial y_1} x_2^{-1}} = \frac{y_0^{d-3} dy_0}{\frac{\partial g}{\partial y_1}} = -\frac{y_0^{d-3} dy_1}{\frac{\partial g}{\partial y_0}}.$$

So

$$(\omega) = (d-3)X \cdot (X_0 = 0) = (d-3)D$$

where D is the hyperplane section $(X_0 = 0)$ of X. In particular,

$$2g - 2 = \deg(\omega) = (d-3)d,$$

so $g = (d-1)(d-2)/2$. (Recall that we proved this earlier for $k = \mathbb{C}$ using the Riemann–Hurwitz formula.)

It remains to prove that $\theta_{d-3}\colon S_{d-3} \to L((d-3)D) = L(K_X)$ is an isomorphism. The map θ_{d-3} is injective because X has degree d so no homogeneous polynomials of degree $d-3$ vanish on X. Now $\dim S_{d-3} = \binom{d-3+2}{2}$ and $l(K_X) = g = (d-1)(d-2)/2$ are equal, so θ_{d-3} is an isomorphism as claimed. $\square$

Remark 23.2. Prop. 23.1 is similar to the following result we obtained earlier for hyperelliptic curves in Ex. 13.2. Let X be a hyperelliptic curve of genus g and $F\colon X \to \mathbb{P}^1$ a degree 2 morphism. Then the affine piece U_0 of X over $(X_0 \neq 0) \subset \mathbb{P}^1$ is given by

$$U_0 = (y^2 = f(x)) \subset \mathbb{A}^2_{x,y} \to \mathbb{A}^1_x$$

where $x = X_1/X_0$ and f is a polynomial in x. Consider the rational 1-form

$$\omega = \frac{dx}{2y} = \frac{dy}{f'(x)}.$$

Then $(\omega) = (g-1)F^*(0:1)$ and $S_{g-1} \to L(K_X)$ is an isomorphism.

Proposition 23.3. *Let $X \subset \mathbb{P}^2$ be a smooth plane curve of degree $d > 1$. Then there exists a map $f\colon X \to \mathbb{P}^1$ of degree $d-1$, but none of smaller degree.*

Proof. If we project from a point $P \in X$ we obtain a map $f\colon X \to \mathbb{P}^1$ of degree $d-1$.

Now suppose $f\colon X \to \mathbb{P}^1$ is a morphism of degree $e \leq d-2$. Let E be a general fibre of f. Then $l(E) \geq 2$ and $\deg E = e$. So the Riemann–Roch formula

$$l(E) - l(K_X - E) = 1 - g + \deg E$$

gives

$$l(K_X - E) \geq g + 1 - e.$$

Recall that $l(K_X) = g$, so this inequality says that the points of E do not impose linearly independent conditions on $l(K_X)$. Now the identification of $|K_X|$ with the set of restrictions of plane curves $C \subset \mathbb{P}^2$ of degree $d-3$ to X given by Prop. 23.1 together with Lem. 23.4 below give a contradiction. $\square$

Lemma 23.4. *Let n be a positive integer. Any $n+1$ distinct points in $\mathbb{P}^2$ impose independent conditions on curves of degree n.*

Proof. Let $p_1, \dots, p_{n+1} \in \mathbb{P}^2$ be $n+1$ distinct points. For $1 \le i \le n$ let l_i be a line through p_i which does not pass through p_{n+1}. Then $C = l_1 + \cdots + l_n$ is a curve of degree n through $p_1, \dots, p_n$ not passing through p_{n+1}. The result follows by induction. □

24 Special divisors

Let X be a smooth projective curve of genus g. Recall the Riemann–Roch formula

$$l(D) - l(K_X - D) = 1 - g + \deg D.$$

We say an effective divisor D on X is *special* if $l(K_X - D) > 0$. Equivalently, let $\phi = \phi_{K_X} : X \to \mathbb{P}^{g-1}$ be the canonical map. Then D is special iff $\phi(D)$ is contained in a hyperplane. (More precisely, being careful about multiplicities, there exists a hyperplane $H \subset \mathbb{P}^{g-1}$ such that $\phi^* H \ge D$.)

Proposition 24.1.
(1) There exist special divisors of degree d for every $1 \le d \le 2g-2$.

(2) Every effective divisor of degree $d \le g-1$ is special.

(3) An effective divisor D of degree g is special iff $l(D) > 1$.

Proof. (1) This follows from $\deg K_X = 2g-2$.

(2) If D has degree $d \le g-1$ then there exists a hyperplane $H \subset \mathbb{P}^{g-1}$ containing $\phi(D)$, so D is special.

(3) If $\deg D = g$ then $l(K_X - D) = l(D) - 1$ by Riemann–Roch, so D is special iff $l(D) > 1$. □

Corollary 24.2. *If $g \ge 2$ then X admits a morphism $F : X \to \mathbb{P}^1$ of degree $\le g$.*

Proof. Let D be a special divisor of degree g and $f \in L(D) \setminus k$, then the morphism $F = (1 : f) : X \to \mathbb{P}^1$ has degree $\le g$. □

Example 24.3. If $g = 3$ then either X is hyperelliptic (that is, admits a degree 2 map to $\mathbb{P}^1$) or X is isomorphic to a plane quartic. In the second case, projecting from a point $P \in X$ defines a degree 3 map to $\mathbb{P}^1$.

We say $P \in X$ is a *Weierstrass point* if gP is special. Equivalently, there exists a hyperplane $H \subset \mathbb{P}^{g-1}$ such that $\phi_{K_X}^* H \geq gP$.

In coordinates, let $\{\omega_0, \ldots, \omega_{g-1}\}$ be a basis of $\Omega_X(X)$, then $\phi = \phi_{K_X}$ is given by $\phi = (\omega_0 : \cdots : \omega_{g-1})$. Let t be a local parameter at $P \in X$ and write $\omega_i = f_i dt$, where $f_i \in \mathcal{O}_{X,P} \subset k[[t]]$. Let $H \subset \mathbb{P}^{g-1}$ be a hyperplane. Then $\phi^* H \geq gP$ iff H contains the points

$$(f_0^{(i)} : \cdots : f_{g-1}^{(i)}), \quad 0 \leq i \leq g-1.$$

(Here $f^{(i)}$ denotes the ith derivative of f with respect to t.) So, there exists H with $\phi^* H \geq gP$ iff the above points are linearly dependent, equivalently, the *Wronskian* $W = \det(f_j^{(i)})$ vanishes.

Let $N = 1 + \cdots + g = g(g+1)/2$. A calculation shows that the $(0, N)$-tensor $\eta = W dt^{\otimes N}$ does not depend on the choice of t. So η defines an global regular $(0, N)$-tensor $\eta \in \Omega_X^{\otimes N}(X)$, and the divisor (η) of zeroes of η is linearly equivalent to NK_X. Thus $\deg(\eta) = N(2g-2) = (g-1)g(g+1)$.

Proposition 24.4. *There are exactly* $(g-1)g(g+1)$ *Weierstrass points on* X *(counted with multiplicities).*

□

Example 24.5. The Weierstrass points of a hyperelliptic curve are the $2g+2$ branch points of the degree 2 map $f\colon X \to \mathbb{P}^1$, each counted with multiplicity $(g-1)g/2$.

The Weierstrass points of a smooth plane quartic are the flexes, and there are $24 = (3-1)(3)(3+1)$ distinct flexes in general.

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